Applied Analysis for Learning Architectures

by

Himanshu Singh

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics & Statistics
College of Arts and Sciences
University of South Florida

Major Professor: Joel A. Rosenfeld, Ph.D.
Benjamin P. Russo, Ph.D.
Catherine Bénéteau, Ph.D.
Sherwin Kouchekian, Ph.D.
Razvan Teodorescu, Ph.D.

Date of Approval:
June 15, 2023

Keywords: Dynamic Mode Decomposition, Liouville, Reproducing Kernel Hilbert space, Koopman, Weighted composition operators, Learning Algorithms

DEDICATION

To,

Mummy and her wish to sit in a flight & Papa and his wish for me becoming an Indian Administrative Officer

ACKNOWLEDGMENTS

I owe my big time to my dear family members and my loved one, *Mr. Manoj Kumar Singh, Akarshi Singh, Drishty Singh & Payal Ghosh*. I also would like to acknowledge the support of *Grandpa - Mr. Gordon Koncelik* during COVID times. Thanks for your unmatched love and support!

This dissertation is my original research work. I take the inspiration of producing quality original work from my PhD Advisor: DR. JOEL A. ROSENFELD. I would like to thanks him and his research support which made my research ideas strong and profound. I also hold special reservation for gratitude to DR. BENJAMIN P. RUSSO and his patience in making my mathematical concepts crystal clear in my struggling days.

I would also like to thanks DR. CATHERINE BÉNÉTEAU, DR. SHERWIN KOUCHEKIAN & DR. RAZVAN TEODORESCU for their valuable time and contribution as my PhD Committee members.

I would like to thanks *Daniel, Ladan, Efrain, John & Chuck* with whom I shared my mathematical insecurities. I must also be thankful to their constant belief in me that one day I'll be a big researcher.

I believe the friendship of Deepak, Rakesh, Devendar and Adhish is precious and has immensely motivated me in this journey. I and my family are specially grateful to DEEPAK and RAKESH.

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ABSTRACT

Modern data science problems revolves around the Koopman operator C_φ (or *Composition operator*) approach, which provides the best-fit linear approximator to the dynamical system by which the dynamics can be advanced under the discretization. The solution provided by Koopman in the data driven methods is in the sense of *strong operator topology*, which is nothing better then the point-wise convergence of data (snapshots) in the underlying Hilbert space. CHAPTER 2 provides the details about the aforementioned issues with essential counter-examples. Thereafter, ***provable convergence guarantee phenomena*** is demonstrated by the Liouville weighted composition operators $A_{f,\varphi}$ over the Fock space by providing the **boundedness** (cf: THEOREM 2.17) and **compactness** (cf: THEOREM 2.25) establishment of $A_{f,\varphi}$ over the Fock Space. Modern learning algorithms such as *support vector machines* and *activation function* for *Neural Nets* heavily relies on the kernel function which arises from the Lebesgue measure and reproducing kernel of Fock space. Thus, this chapter serves its related results by establishing the *norm convergence* which benefits the practitioners of learning algorithms community.

CHAPTER 3 investigates the interaction of the weighted composition operators over the Paley-Wiener Space. For sampling and interpolation of signals, Paley-Wiener Space is crucial. **The distinctive contribution of this chapter is the discussion of the *Phragmén-Lindelöf indicator function* and the *Pólya representation* in the presence of the weighted composition operators over the Paley-Wiener Space**. Additionally, it has been demonstrated that there exist no compact composition operators over the Paley-Wiener Space. However, **due to the presence of an extra symbol (*multiplication-* ψ) in the**

weighted composition operator, one can leverage it to attain the compactness of it over the Paley-Wiener Space.

CHAPTER 4 introduce the Koopman or composition operator over the Poly-logarithmic Hardy Space. As a result of this, we also learned about the *Nevanlinna Counting Function* for the Poly-logarithmic Hardy Space.

CHAPTER 5 provides the theory of Mittag-Leffler Space of entire functions in the setting of L^p space. **The chief contribution of this chapter are the *duality, isometry, boundedness of integral operators* via the Schur's test which leads to perform the *atomic decomposition* of the Mittag-Leffler Space**.

CHAPTER 1

INTRODUCTION

The research progress made in the present dissertation sits in the center of the theory of *Reproducing Kernel Hilbert Spaces* (RKHS) and *Operator Theory.*

Definition 1.1 (cf. [1])**.** *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and $X \neq \emptyset$. Then H is defined as a $\mathbb{K}$-reproducing kernel Hilbert space comprising if for functions $f : X \to \mathbb{K}$ such that*

1. *A function $K : X \times X \to \mathbb{K}$ is called a reproducing kernel of H if $K(\cdot, x) \in H$ $\forall x \in X$ and the reproducing property holds i.e.*

$$f(x) = \langle f, K(\cdot, x)\rangle_H, \ \forall f \in H \text{ and } x \in X. \tag{RK}$$

2. *H is called RKHS over X if $\forall x \in X$, the point evaluation functional $\mathfrak{P}_x : H \to \mathbb{K}$ defined by $\mathfrak{P}_x(f) \coloneqq f(x)$ for $f \in H$ is continuous.*

The existence of the reproducing kernel $K(\cdot, x)$ in (RK) is ensured by the Riesz Representation theorem; states as follows:

Theorem 1.2 (Riesz Representation Theorem)**.** *Let H be a Hilbert Space and let $L : H \to \mathbb{K}$ be a bounded linear functional. Then $\exists$ a unique vector $h_0 \in H$ such that $L(h) = \langle h, h_0\rangle_H$, $\forall h \in H$. Furthermore, $\|L\|_H = \|h_0\|_H$.*

1.1 Examples of RKHS

The examples of RKHS for entire functions are provided here. These examples will play an important roles in this dissertation.

1.1.1 Fock Space

Consider the *Gaussian measure* as $d\mathbf{A}_{\mathbb{G}} = \dfrac{1}{\pi^n} e^{-|z|^2} dA(z)$ for $z \in \mathbb{C}^n$, where dA is ordinary Lebesgue measure over $\mathbb{C}^n$. Consider $L^2(\mathbb{C}^n, d\mathbf{A}_{\mathbb{G}})$ as the Lebesgue measurable space comprising of measurable functions g for which the following holds:

$$\|g\|^2_{\mathbf{F}^2(\mathbb{C}^n)} := \int_{\mathbb{C}^n} |g(z)|^2 d\mathbf{A}_{\mathbb{G}}(z) < \infty. \tag{1.1}$$

The Fock space, $\mathbf{F}^2(\mathbb{C}^n)$, is given as the subset of $L^2(\mathbb{C}^n, d\mathbf{A}_{\mathbb{G}})$ whose elements have an entire function as a representative of their equivalence class. Since analyticity is preserved through convergence with respect to the $L^2(\mathbb{C}^n, d\mathbf{A}_{\mathbb{G}})$ norm, $\mathbf{F}^2(\mathbb{C}^n)$ is closed, and is identified as a space of entire functions. $\mathbf{F}^2(\mathbb{C}^n)$ is a reproducing kernel Hilbert space, see [53], with the kernel given as

$$K_w(z) = K(z, w) = e^{\langle z,w\rangle} = e^{z\cdot\overline{w}}, \quad z,\ w \in \mathbb{C}^n, \tag{1.2}$$

and the normalized kernel function at $w \in \mathbb{C}^n$ for $\mathbf{F}^2(\mathbb{C}^n)$ is given as

$$k_w(z) = \exp\left(\langle z, w\rangle - \frac{|w|^2}{2}\right), \quad z \in \mathbb{C}^n. \tag{1.3}$$

1.1.2 Mittag-Leffler Space

Recall the one-parameter Mittag-Leffler's classical function; that is defined by the power series given as follows:

$$E_q(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(qk+1)},\ \text{where } \Gamma(\tau)=\int_0^{\infty}t^{\tau-1}e^{-\tau}dt. \tag{1.4}$$

The Mittag-Leffler space, $ML^2(\mathbb{C};q)$, of entire functions of order $q>0$, is defined as follows:

$$ML^2(\mathbb{C};q):=\left\{f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n:\sum_{n=0}^{\infty}|\hat{f}(n)|^2\Gamma(qn+1)<\infty\right\},$$

We define the norm of $f\in ML^2(\mathbb{C};q)$ as follows:

$$\|f\|^2:=\int_{\mathbb{C}}|f(z)|^2d\mu_q(z),\qquad \textit{where}, \tag{1.5}$$

$$d\mu_q(z):=\frac{1}{q\pi}|z|^{\frac{2}{q}-2}e^{-|z|^{\frac{2}{q}}}dA(z). \tag{1.6}$$

Remark 1.3. *Note that if $q=1$ in (1.6), then $d\mu_q(z)$ is the usual* Gaussian *measure (see [53, Chapter 2, Page 33]). For additional and particular details about the norm, measure of $ML^2(\mathbb{C};q)$ given in (1.5) and (1.6) and its connection with the Fock space, follow [34].*

Proposition 1.4 (cf. [34] and [18].). *Following properties holds for $ML^2(\mathbb{C};q)$:*

1. *The set $\{\mathbf{e}_j\}_{j=0}^{\infty}$ in (1.7), forms the orthonormal basis for $ML^2(\mathbb{C};q)$*

$$\{\mathbf{e}_j(z)\}_j=\left\{\frac{z^j}{\sqrt{\Gamma(qj+1)}}\right\}_j. \tag{1.7}$$

2. *For all $f\in ML^2(\mathbb{C};q)$, the following inequality is enjoyed by f,*

$$|f(z)|^2\leq E_q(|z|^2)\|f\|^2. \tag{1.8}$$

For $q > 0$, we define the *normalized reproducing kernel* k_z at $z \in \mathbb{C}$ for $ML^2(\mathbb{C}; q)$ as follows:

$$k_z = \frac{K_z}{\|K_z\|}, \tag{1.9}$$

where $\|K_z\| = \sqrt{K(z,z)} = \sqrt{E_q(|z|^2)}$.

Lemma 1.5. *The normalized reproducing kernel of $ML^2(\mathbb{C}; q)$ given* (1.9)*, is a unit vector in $ML^2(\mathbb{C}; q)$.*

Proof. The proof is easy once we use the reproducing property of the reproducing kernel K_λ, that is, $\langle f, K_\lambda \rangle = f(\lambda)$ in $ML^2(\mathbb{C}; q)$. Now, we give the proof for this as follows:

$$\begin{aligned}
\|k_\lambda\|^2 &= \frac{1}{q\pi} \int_{\mathbb{C}} |k_\lambda(z)|^2 |z|^{\frac{2}{q}-2} e^{-|z|^{\frac{2}{q}}} dA(z), \\
&= \frac{1}{q\pi} \int_{\mathbb{C}} \left| \frac{K_\lambda(z)}{\sqrt{K_\lambda(\lambda)}} \right|^2 |z|^{\frac{2}{q}-2} e^{-|z|^{\frac{2}{q}}} dA(z), \\
&= \frac{1}{q\pi K_\lambda(\lambda)} \int_{\mathbb{C}} K_\lambda(z) K_z(\lambda) |z|^{\frac{2}{q}-2} e^{-|z|^{\frac{2}{q}}} dA(z), \\
&= 1.
\end{aligned}$$

Therefore, k_λ is a unit vector in $ML^2(\mathbb{C}; q)$, as desired. $\square$

1.1.3 Paley-Wiener Space

Definition 1.6. *An entire function f is of exponential type τ if $|f(z)| \leq O\left(\exp\left((\tau + \epsilon)|z|\right)\right)$ $\forall \epsilon > 0$.*

We will use either τ or σ symbols are reserved for 'type' for a function which is an *entire function of exponential type* $\mathcal{EFET}$, in this dissertation. Recall the classic result of Paley-Wiener theorem given as:

Theorem 1.7 (Paley and Wiener)**.** *Suppose $f \in L^2(\mathbb{R})$. Then the necessary and sufficient condition that the Fourier transform of f vanish outside of $[-\sigma, \sigma]$ is that $f(x)$ be of exponential type σ.*

Paley-Wiener space, PW_π^2, is comprised of entire functions of exponential type σ less than or equal to π whose restriction to the real line belongs to the space $L^2(\mathbb{R})$, that is, entire function f with $|f(z)| \leq O(\exp((\sigma+\epsilon)|z|))$ for all $\epsilon > 0$, and for all $\sigma \leq \pi$, satisfies the following:

$$\|f\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |f(x)|^2 dx < \infty, \qquad \forall\, f \in PW_\pi^2. \tag{1.10}$$

The aforementioned definition of PW_π^2 is the result of Theorem 1.7. PW_π^2 is a RKHS whose norm $\|\bullet\|_{PW_\pi^2}$ is given as (1.10), that is, $\|f\|_{PW_\pi^2} \equiv \|f\|_{L^2(\mathbb{R})} < \infty$. For $f \in PW_\pi^2$ and with $z = x + iy$, the *Plancherel-Pólya theorem* states that $\int_{\mathbb{R}} |f(z)|^2 dx \leq \|f\|_{PW_\pi^2}^2 e^{2\pi|\operatorname{Im} z|}$ for f being an $\mathcal{EFET}$ since $\sigma \leq \pi$ [27, Page-51, Theorem 4]. The reproducing kernel of PW_π^2 is $K(z, w) = \operatorname{sinc}(z - \overline{w}) = \dfrac{\sin(\pi(z-\overline{w}))}{\pi(z-\overline{w})}$. For $n \in \mathbb{Z}$, $\{K_n\}_n$ forms the orthonormal basis of PW_π^2, since $\operatorname{sinc}(z-n)$ is the image of $\dfrac{e^{-int}}{\sqrt{2\pi}}$ under the inverse Fourier transform. For $f \in PW_\pi^2$, the relationship $\|f\|_{L^2(\mathbb{R})}^2 = \sum_{n\in\mathbb{Z}} \|f(n)\|^2$ is true by the virtue of *Parseval's identity*. Every candidate $f \in PW_\pi^2$ enjoys the following representation against $\hat{f} \in L^2(-\pi, \pi)$:

$$f(z) = \int_{-\pi}^{\pi} \hat{f}(t) e^{itz} \frac{dt}{\sqrt{2\pi}}.$$

There exists a unitary Fourier isometric isomorphic transform from $PW_\pi^2 \to L^2(-\pi, \pi)$, and again as an application of Parseval's identity we have

$$\|f\|_{L^2(\mathbb{R})}^2 = \|\hat{f}\|_{L^2(-\pi,\pi)}^2. \tag{1.11}$$

Let $f \in PW_\pi^2$. Recall [27, Lecture-10, Page-69] as follows:

$$|f(x+iy)| \leq \frac{e^{-\pi|y|}}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(t)| dt \tag{1.12}$$

$$|f(x+iy)|^2 \leq \frac{e^{-2\pi|y|}}{2\pi} \|f\|_{PW_\pi^2}^2, \qquad (\textit{use Cauchy-Schwarz}\ \&\ (1.11)). \tag{1.13}$$

We now include important examples of RKHS over a finite measure space, in particular the unit disc $\mathbb{D}$.

1.1.4 Hardy Space

The Hardy space, $H^2(\mathbb{D})$, is a Hilbert space comprised of all holomorphic functions f in $\mathbb{D}$ for which the following holds:

$$\|f\|_{H^2}^2 = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty. \tag{1.14}$$

Upon the calculation of (1.14) for $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, the following summation correspondence holds true for $f \in H^2$,

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty. \tag{1.15}$$

1.1.4.1 Littlewood-Paley Identity for H^2 We need following elementary equality that can be easily proved by integration by parts technique.

$$\int_0^1 r^{2n-1} \log \frac{1}{r} dr = \frac{1}{4n^2}, \qquad n \in \mathbb{Z}_+. \tag{1.16}$$

The *Littlewood-Paley Identity* for H^2 is given above and we give the proof of it for clarity.

$$\|f\|_{H^2}^2 = |f(0)|^2 + \frac{2}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z), \qquad \textit{for } f \in H^2. \tag{1.17}$$

Proof of (1.17). The *LHS* of (1.17) is $\sum_{n=0}^{\infty}|\hat{f}(n)|^2$. On the other hand,

$$\begin{aligned}
\int_{\mathbb{D}}|f'(z)|^2\log\frac{1}{|z|}dA(z) &= \int_{\mathbb{D}}\left|\sum_{n=1}^{\infty}n\hat{f}(n)z^{n-1}\right|^2\log\frac{1}{|z|}dA(z),\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nm\hat{f}(n)\overline{\hat{f}(m)}2\pi\delta_{nm}\int_0^1 r^{n+m-1}\log\frac{1}{r}dr,\\
&= 2\pi\sum_{n=1}^{\infty}n^2|\hat{f}(n)|^2\int_0^1 r^{2n-1}\log\frac{1}{r}dr, \quad (\textit{use}\ (1.16)),\\
&= \frac{\pi}{2}\sum_{n=1}^{\infty}|\hat{f}(n)|^2.
\end{aligned}$$

Therefore, $|f(0)|^2+\frac{2}{\pi}\int_{\mathbb{D}}|f'(z)|^2\log\frac{1}{|z|}dA(z) = |\hat{f}(0)|^2+\sum_{n=1}^{\infty}|\hat{f}(n)|^2 = \|f\|_{H^2}^2$. □

Remark 1.8. *The Littlewood-Paley Identity for H^2 is useful for instance, as one can learn from* (1.17) *that for $f \in H^2$, its norm is equivalent to the condition that $f' \in L^2\left(\mathbb{D}, \frac{2}{\pi}\log\frac{1}{|z|}dA(z)\right)$ up-to the addition of $|f(0)|^2$. In other words, here $\frac{2}{\pi}\log\frac{1}{|z|}$ is the right measure that matches with the derivative of function in H^2 in L^2 setting over $\mathbb{D}$. The similar phenomena will rise for different Hilbert space of entire functions. However, the name of* equivalent norm representation, *abbreviated as ENR will be utilized for the discussed Hilbert space. Therefore, the Littlewood-Paley Identity is* only *associated with H^2.*

Let us now define the *Polylogarithmic Hardy space* as follows:

1.1.5 Polylogarithmic Hardy Space

Definition 1.9. *Let $s \in \mathbb{C}$. The Polylogarithmic Hardy space $PL^2(\mathbb{D}; s)$ is defined as follows:*

$$PL^2(\mathbb{D}; s) = \left\{ f_s(z) = \sum_{k=1}^{\infty}\frac{\hat{f}(k)}{k^s}z^k : \sum_{k=1}^{\infty}\left|\hat{f}(k)\right|^2 < \infty \right\}. \tag{1.18}$$

Note that $f_s \in PL^2(\mathbb{D}; s)$ is entire in s and analytic for $|z|< 1$ (cf. [33]). The fact that f_s is analytic implies the existence of *holomorphic* character of f_s. $PL^2(\mathbb{D}; s)$ is a reproducing kernel Hilbert space with the kernel function $K(z, w; s, t)$ given as:

$$K(z, w; s, t) = L_{s+t}(z\overline{w}), \text{ where } L_s(z) = \sum_{n=0}^{\infty} z^n \cdot n^{-s}$$

for s and $t \in \mathbb{R}$. Consider f_s and g_s in $PL^2(\mathbb{D}; s)$, then that inner product for the same is defined in relation with H^2 as:

$$\begin{aligned} \langle f_s, g_s, \rangle_{PL^2(\mathbb{D};s)} &= \langle S^* f_0, S^* g_0 \rangle_{H^2}, \\ &= \frac{1}{2\pi} \int_0^{2\pi} S^* f_0(e^{i\theta}) \overline{S^* g_0(e^{i\theta})} d\theta, \end{aligned} \tag{1.19}$$

where $S = M_z$ is the shift operator. In addition to this, the norm of $f_s \in PL^2(\mathbb{D}; s)$ is:

$$\|f_s\|^2_{PL^2(\mathbb{D};s)} = \frac{1}{2\pi} \int_0^{2\pi} |S^* f_0(e^{i\theta})|^2 d\theta < \infty. \tag{1.20}$$

1.1.6 Dirichlet Space

The Dirichlet space, $D^2(\mathbb{D})$, is a Hilbert space comprised of all holomorphic functions f in $\mathbb{D}$ for which the following holds:

$$\|f\|^2_{D^2} = |f(0)|^2 + \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty, \text{where } dA \text{ is the area measure.} \tag{1.21}$$

Note that for $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ with the calculation of (1.21), the following summation correspondence holds true for $f \in D^2$,

$$\|f\|^2_{D^2} = \sum_{n=0}^{\infty} (n+1) |\hat{f}(n)|^2 < \infty. \tag{1.22}$$

1.1.7 Bergman Space

The Bergman space, $A^2(\mathbb{D})$, is a Hilbert space comprised of all holomorphic functions f in $\mathbb{D}$ for which the following holds:

$$\|f\|_{A^2}^2 = \frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty. \tag{1.23}$$

For $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in A^2$:

$$\|f\|_{A^2}^2 = \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} < \infty. \tag{1.24}$$

The following space containment relation among all three Hilbert spaces, is true:

$$D^2 \subset H^2 \subset A^2. \tag{1.25}$$

Now, we revisit some modern theory of weighted composition operators over various RKHS.

1.2 Weighted composition operators

Let H be a Hilbert space of entire functions over $\mathbb{C}$. Let ψ and φ be entire functions.Then, the *weighted composition operator* denoted formally by $W_{\psi,\varphi}$ is defined as

$$W_{\psi,\varphi} f := \psi \cdot (f \circ \varphi) \ \ \textit{under} \ W_{\psi,\varphi} : \mathrm{dom}\,(W_{\psi,\varphi}) \to H.$$

The $\mathrm{dom}\,(W_{\psi,\varphi})$ is the domain of $W_{\psi,\varphi}$ such that $\psi \cdot (f \circ \varphi) \in H$. For $\psi = 1$, $W_{\psi,\varphi}$ is the classical *Composition operators.*

The history of $W_{\psi,\varphi}$ traces back to 1964 when Forelli demonstrated the isometries among the Hardy spaces H^p for $1 < p < \infty$ and $p \neq 2$, see [11].

1.2.1 Disjoint Support Property

Let X be a Hausdorff topological space and let E be a real or complex Banach space. The space of continuous E valued functions on X will be denoted by $C(X, E)$. This space is a Banach space when endowed with the usual norm

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

Disjoint support property of an operator T on $C(X, E)$ is defined then as:

$$\boxed{\text{if } \|f(x)\|\|g(x)\| = 0 \text{ for every } x \in X \implies \|Tf(x)\|\|Tg(x)\| = 0 \text{ for every } x \in X.}$$

It has been demonstrated that the only bounded operator over $C(X, E)$ which satisfy with the disjoint support property are precisely the weighted composition operators $W_{\psi,\varphi}$; which was shown by Jamison and Rajagopalan in [21].

1.2.2 Results by Carswell, MacCluer and Schuster

Carswell, MacCluer and Schuster characterized bounded and compact C_φ over the Fock space $\mathbf{F}^2(\mathbb{C}^n)$ of entire functions [4].

Theorem 1.10 (Theorems-1 & 2, [4])**.** *Composition operator C_φ is **bounded** on $\mathbf{F}^2(\mathbb{C}^n)$ only when $\varphi(z) = Az + B$ with $\|A\| \leq 1$. Conversely, suppose that $\varphi(z) = Az + B$ with A being a $n \times n$ complex matrix and $B \in \mathbb{C}^n$. If $\|A\| = 1$ then C_φ is bounded on $\mathbf{F}^2(\mathbb{C}^n)$ if and only if $B = 0$. If $\|A\| < 1$ then C_φ acts **compactly** on $\mathbf{F}^2(\mathbb{C}^n)$.*

1.2.3 Results by Ueki and Le

For this particular section, we will use UEKI's notation: $uC_\varphi f$ to denote the weighted composition operator. Study of $uC_\varphi f := u \cdot f \circ \varphi$ with certain integral transforms denoted by $B_\varphi(|u|^2)(w)$ in [49] by UEKI. The quantity $B_\varphi(|u|^2)(w)$ is given as follows:

$$B_\varphi(|u|^2)(w) = \frac{1}{2\pi}\int_{\mathbb{C}} |u(z)|^2 \left| e^{\frac{\langle \varphi(z), w\rangle}{2}} \right|^2 e^{-\frac{|w|^2}{2}} e^{-\frac{|z|^2}{2}} dA(z).$$

The boudedness and compactness characterization of $uC_\varphi f$ over the Fock space (for a single variable, $\mathcal{F}^2$) by UEKI is given as follows:

Theorem 1.11. *Follow [49, Theorem 1 & Theorem 2] respectively for following results.*

1. *uC_φ is a bounded operator on $\mathcal{F}^2$ if and only if $B_\varphi(|u|^2) \in L^\infty(\mathbb{C})$.*

2. *Suppose uC_φ is bounded over $\mathcal{F}^2$, then there is a positive constant C such that*

$$\limsup_{|w|\uparrow\infty} B_\varphi(|u|^2)(w) \leq \|uC_\varphi\|_{\mathrm{e}} \leq C \limsup_{|w|\uparrow\infty} B_\varphi(|u|^2)(w).$$

In particular, uC_φ is a compact operator on $\mathcal{F}^2$ if and only if $\lim_{|w|\uparrow\infty} B_\varphi(|u|^2)(w) = 0$.

Related results were further-up simplified in [26] by LE.

Theorem 1.12. *Follow [26, Theorem 2.2. & Theorem 2.4.] respectively for following results.*

1. *$W_{f,\varphi}$ is bounded on $\mathcal{F}^2$ if and only if f belongs to $\mathcal{F}^2$, $\varphi(z) = \varphi(0) + \lambda z$ with $|\lambda| \leq 1$ and*

$$M(f,\varphi) := \sup |f(z)|^2 \exp\left(|\varphi(z)|^2 - |z|^2\right) : z \in \mathbb{C}\} < \infty.$$

2. *$W_{f,\varphi}$ is compact on $\mathcal{F}^2$ if and only if $\varphi(z) = \varphi(0) + \lambda z$ for some $|\lambda| < 1$ and*

$$\lim_{|z|\uparrow\infty} |f(z)|^2 \exp(|\varphi(z)|^2 - |z|^2) = 0$$

holds.

1.3 Liouville operators and Occupation Kernels

ROSENFELD, RUSSO and KAMALAPURKAR et.al. studied *Dynamic Mode Decomposition* (DMD) for continuous dynamical systems in the setting *Liouville operators* paired up with *Occupation kernels* [35, 36, 37, 42]. Further, occupation kernels have played a key role to study the motion tomography in [43] and fractional order system identification [28].

Definition 1.13 (Liouville Operators). *Let $\dot{\gamma} = f(\gamma)$ be a dynamical system with $f : \mathbb{R}^n \to \mathbb{R}^n$ as a locally Lipschitz function. Let H be a RKHS over compact $X \subset \mathbb{R}^n$, then Liouville operator native to the given dynamics $A_f : \mathbf{D}(A_f) \to H$ operates as*

$$A_f g(*) := \nabla g(*) \cdot f(*) \text{ where } \mathbf{D}(A_f) := \{h \in H \mid \nabla h \cdot f \in H\}. \tag{1.26}$$

Fruitful results are discovered when A_f is coupled with *Occupation Kernel*, defined as follows:

Definition 1.14 (Occupation Kernel). *Take a continuous trajectory $\gamma : [0, T] \to X \subset \mathbb{R}^n$ where X is compact in $\mathbb{R}^n$. Let $(H, \langle *, * \rangle_H)$ be the RKHS over X consisting of continuous functions; the functional $g \mapsto \int_0^T g(\gamma(\tau))d\tau$ is bounded. Appealing to the Riesz Representation theorem,*

$$\int_0^T g(\gamma(\tau))d\tau = \langle g, \Gamma_\gamma \rangle_H$$

for some $\Gamma_\gamma \in H$. This Γ_γ is called as an occupation kernel *corresponding to $\gamma \in H$.*

Letting $K(\cdot, \cdot)$ be the kernel function corresponding to H, then following holds:

$$A_f^* \Gamma_\gamma = K(\cdot, \gamma(T)) - K(\cdot, \gamma(0)).$$

$A_f^* \Gamma_\gamma$ provides the input-output relationships that enable DMD of time series data. Liouville operators in a DMD routine allows the dynamics to be locally rather than globally Lipschitz. $\{A_f\}$ when examined via $\{\Gamma_\gamma\}$ avoids the limiting relations of Koopman operators that might not be well defined for a particular discretization of a continuous time nonlinear dynamical system. Readers are encouraged to follow important references for this: [36, 38].

CHAPTER 2
PROVABLE CONVERGENCE GUARANTEE SOLUTION FOR LEARNING ALGORITHMS

Since the beginning of the 21st century, there has been a tremendous *data* production, of which a majority of it is unprocessed and unanalysed which eventually is unused. Giant domains where the data misses the opportunity of even being understood are commerce, technology, health and network sectors. Often times, these data are affiliated with certain dynamical systems which are nonlinear and chaotic. To understand and process the generated data, Koopman operator framework is employed.

2.1 Koopman Operator

2.1.1 Forward Complete Dynamics

Koopman operator framework or mathematically just the *Composition operator* has emerged as one of the main candidates for machine learning of dynamical processes. In particular, practitioners employ C_φ in the sense of *Koopman* [24] to deal with the discretizable dynamical systems and Dynamic Mode Decomposition (DMD) [15, 25, 29, 50] and etc. Koopman operator technique/analysis fails to capture the data-driven analysis for the dynamics which are not discretizable, for instance $\dot{x} = 1 + x^2$. Here, for a fixed and finite time-incremental δt in $\left(0, \frac{\pi}{2}\right)$ the corresponding discrete time dynamics are $x_{m+1} = \tan\left(\arctan(x_m) + \delta t\right)$. Observe that when x_m takes $\tan\left(\frac{\pi}{2} - \delta t\right)$, it results into the absurdity of x_{m+1}. Hence, the symbol φ to the corresponding Koopman operator is not well-defined. The example discussed here is one of the simplest example belonging to the polynomial dynamics.

These dynamics are prevalent in the literature of mass-action kinetics in thermodynamics, chemical reactions and species populations. These situation makes the study of Liouville operators as a big industry in various themes.

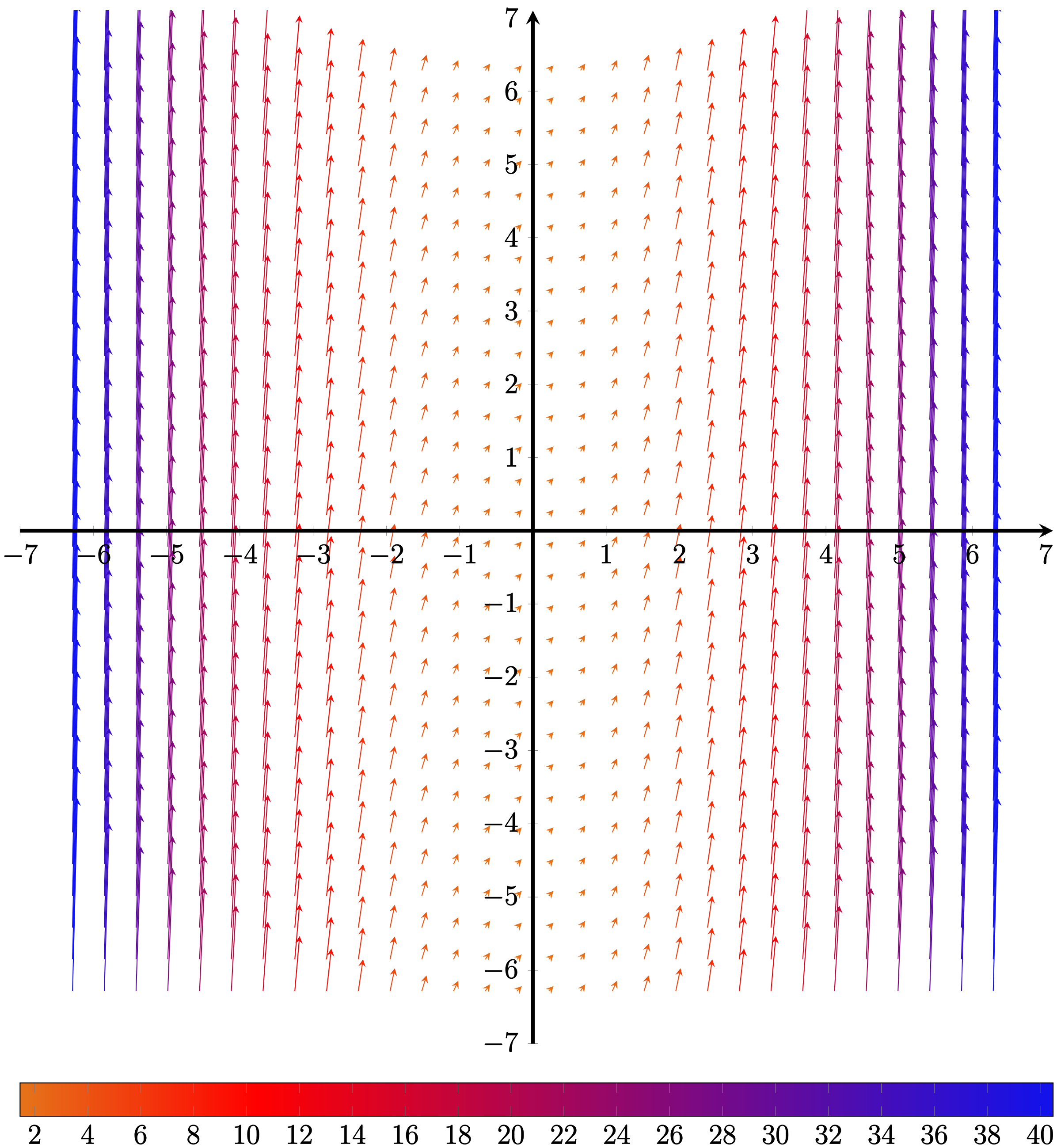

Figure 1. Vector field of dynamical system $\dot{x}(t) = 1 + x^2(t)$

2.1.2 Affine Symbols

2.1.2.1 For the Fock Space A proof-sketch of THEOREM 1.10 is addressed here for the Koopman operators C_φ interacting on the Fock Space $\mathbf{F}^2(\mathbb{C}^n)$.

Proof sketch of THEOREM 1.10. If C_φ is *bounded* over $\mathbf{F}^2(\mathbb{C}^n)$ then

$$\sup_{w\in\mathbb{C}^n} \frac{\|C_\varphi^*(k_w)\|}{\|k_w\|} = \sup_{w\in\mathbb{C}^n} \exp\left(\frac{1}{4}\left(|\varphi(w)|^2 - |w|^2\right)\right) < \infty.$$

From above, we infer that

$$\limsup_{|w|\uparrow\infty} \frac{|\varphi(w)|}{|w|} \leq 1. \tag{2.1}$$

Thus, each coordinate of φ is linear, that is, $\varphi(z) = Az + B$. Besides this, say $|A\zeta| > |\zeta|$ for some $\zeta \in \mathbb{C}^n$ whose norm is 1. Letting $z = \tau\zeta$, where $\tau > 0$ and then making $\tau \to \infty$ and combined with (2.1) as follows:

$$\lim_{\tau\to\infty} \frac{|A\zeta + \frac{1}{\tau}B|}{|\zeta|} < 1.$$

The above inequality implies $|A\zeta| < |\zeta|$ which is a contradiction. For *compactness*, let $\|A\| = 1$ and note that

$$\|C_\varphi\|_{\mathrm{e}} \geq \limsup_{|w|\uparrow\infty} \exp\left(\frac{1}{4}\left(|\varphi(w)|^2 - |w|^2\right)\right),$$

which allows $\|C_\varphi\|_{\mathrm{e}} \neq 0$, so C_φ is not compact. Thus, $\|A\| < 1$. $\square$

2.1.2.2 For the Paley-Wiener Space Fundamental to the data sampling and data interpolation is the classical space PW_{π}^{2}. We learned from SUBSECTION 1.1.3 that it comprises of only those entire functions of exponential type whose restriction of it on real axis is $L^2(\mathbb{R})$-integrable. It has been demonstrated in [5] that the only Koopman operators that can act bounded over PW_{π}^{2} are affine. Moreover, aforementioned reference also proved that there exist no compact Koopman operators for PW_{π}^{2}.

2.2 Preliminaries

Remark 2.1. *Unless otherwise stated, all vectors $v \in \mathbb{C}^n$ are column vectors, $v^\top$ denotes the transpose of a vector or matrix. When $n = 1$, the symbols $|\cdot|$ implies the modulus of a complex number. The symbols $\|\bullet\|$ will represent either $\mathbf{F}^2(\mathbb{C}^n)$ norm (defined in (1.1)) or operator norm depending on the context.* $\det(\mathcal{A})$ *denotes the determinant of a matrix $\mathcal{A}$. The multi-index notation is employed as $j = (j_1, \ldots, j_n)$ and $|j| = j_1 + \cdots + j_n$. If $z = (z_1, \ldots, z_n)$ then $z^j = (z_1^{j_1}, \ldots, z_n^{j_n})$, and $dz = dz_1 \cdots dz_n$.*

Definition 2.2. *Let $\mathbf{D}(A_{f,\varphi})$ be a subspace of $\mathbf{F}^2(\mathbb{C}^n)$, $A_{f,\varphi} : \mathbf{D}(A_{f,\varphi}) \to \mathbf{F}^2(\mathbb{C}^n)$, then the Liouville weighted composition operator induced by entire multiplication symbol f and an entire composition symbol φ with $f : \mathbb{C}^n \to \mathbb{C}^n$ and $\varphi : \mathbb{C}^n \to \mathbb{C}^n$, is given as follows:*

$$A_{f,\varphi} g \coloneqq \nabla g(\varphi(\cdot)) \cdot D\varphi(\cdot) \cdot f(\cdot), \quad \text{where} \tag{2.2}$$

$$\mathbf{D}(A_{f,\varphi}) \coloneqq \left\{ g \in \mathbf{F}^2(\mathbb{C}^n) \text{ such that } \nabla g(\varphi(\cdot)) \cdot D\varphi(\cdot) \cdot f(\cdot) \in \mathbf{F}^2(\mathbb{C}^n) \right\}.$$

The term '∇g' implies the usual gradient *of g and $D\varphi$ implies the matrix-valued derivative taken on φ.*

2.2.1 Simple Preliminaries

Lemma 2.3. *Let $\varphi : \mathbb{C}^n \to \mathbb{C}$ be holomorphic on a domain Ω containing the closed unit ball. If $\varphi(z) = \sum_{|j|=0} a_j z^j$, then*

$$\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|\varphi|^2 d\theta = \sum_{|j|=0}|a_j|^2 r^{2j}, \qquad a_j \in \mathbb{C}.$$

Jensen's Inequality will also be utilized in this manuscript, and it is stated here for clarity.

Proposition 2.4. *(Jensen's Inequality, [12, Lemma 6.1, Page 33]) Let (Ω, Σ, μ) be a probability space, and g a real-valued function that is μ-integrable. If ψ is a convex function then,*

$$\psi\left(\int_\Omega g\, d\mu\right) \leq \int_\Omega \psi \circ g\, d\mu.$$

Theorem 2.5 (Cauchy's integral formula for distinguished boundary)**.** *Let $f(z)$ be analytic function of several complex variables $z = (z_1, \cdots, z_n)$ in a closed polydisc $\overline{\mathcal{D}}$, $\mathcal{D} = \{z \in \mathbb{C}^n : |z_k - a_k| < r_k\}$. Then, at each point of $\mathcal{D}$, $f(z)$ is representable by a multiple Cauchy integral:*

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\mathbf{T}} \frac{f(\zeta)}{(\zeta - z)} d\zeta, \tag{2.3}$$

where

$$\mathbf{T} = \{\zeta \in \mathbb{C}^n : |\zeta_k - a_k| = r_k,\ k = 1, \ldots, n\}$$

is the distinguished boundary of the polydisc, $\zeta = (\zeta_1, \ldots, \zeta_n)$, $d\zeta = d\zeta_1 \cdots d\zeta_n$ and $\zeta - z = (\zeta_1 - z_1)\cdots(\zeta_n - z_n)$.

Proof. Follow [20, Page 553, Equation 17], or [13, Page 11, Theorem 12]. □

2.2.2 Closeness

The following theorem establishes that Liouville weighted composition operators are closed operators. Hence, when the domain of the operators encompass the entire space, Liouville weighted composition operators are bounded by the closed graph theorem.

Theorem 2.6. *Let H be a reproducing kernel Hilbert space of continuously differentiable functions over $\mathbb{C}^n$. Consider entire mappings $f : \mathbb{C}^n \to \mathbb{C}^n$ and $\varphi : \mathbb{C}^n \to \mathbb{C}^n$. Then $A_{f,\varphi}$ is a closed operator.*

Proof. Suppose $\{g_m\}_{m=1}^{\infty} \subset \mathbf{D}(A_{f,\varphi})$ such that $\|g_m - g\|_H \to 0$ in H. It follows that $\left\{\frac{\partial g_m}{\partial z_i}(z)\right\}$ converges uniformly to $\frac{\partial g}{\partial z_i}(z)$ on compact subsets of $\mathbb{C}^n$ (cf. [48, Corollary 4.36]). Suppose also that $A_{f,\varphi} g_m \to h$ for some $h \in H$, then

$$\nabla g_m(\varphi(z))D(\varphi(z))f(z) \xrightarrow{\text{pointwise}} \nabla g(\varphi(z))D(\varphi(z))f(z).$$

Since, $A_{f,\varphi} g_m(z) = \nabla g_m(\varphi(z))D(\varphi(z))f(z)$, it follows that

$$h(z) = \lim_{m\to\infty} A_{f,\varphi} g_m(z) = \lim_{m\to\infty} \nabla g_m(\varphi(z))D(\varphi(z))f(z) = \nabla g(\varphi(z))D(\varphi(z))f(z).$$

Thus, $\nabla g(\varphi)D(\varphi)f = h$ and $g \in \mathrm{dom}(A_{f,\varphi})$. $\square$

Note that the Theorem 2.6 generalizes of [39, Theorem 3.3].

2.2.3 Liouville weighted composition operators & Occupation Kernels

Recall the definition of Occupation kernels from Definition 1.14. The interaction of $A_{f,\varphi}$ on the occupation kernels is given as follows:

Theorem 2.7. *Let H be a RKHS of continuously differentiable functions with $K(\cdot,\cdot)$ as its kernel. Let γ be a continuous trajectory, $\gamma : [0, T] \to \mathbb{R}^n$ satisfying $\dot{\gamma} = f(\gamma)$ where*

$f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous and $g \to \langle A_{f,\varphi} g, \Gamma_\gamma \rangle_H$ bounded, then for $\Gamma_\gamma \in \mathbf{D}(A^*_{f,\varphi})$,

$$A^*_{f,\varphi}\Gamma_\gamma = K(\cdot, \varphi(\gamma(T))) - K(\cdot, \varphi(\gamma(0))).$$

Proof. It can be proved as follows:

$$\begin{aligned}
\langle A_{f,\varphi} g, \Gamma_\gamma \rangle_H &= \int_0^T A_{f,\varphi} g(\gamma(t)) dt \\
&= \int_0^T \frac{d}{dt} \left(g \circ \varphi(\gamma(t))\right) dt \\
&= \int_0^T \nabla g(\varphi(\gamma(t))) D(\varphi(\gamma(t)))\dot{\gamma}(t) dt, \\
&= g(\varphi(\gamma(T))) - g(\varphi(\gamma(0))), \qquad\qquad (\textit{Fundamental theorem of Calculus}) \\
&= \langle g, K\left(\dagger, \varphi(\gamma(T))\right) - K(\dagger, \varphi(\gamma(0))) \rangle_H, \\
\therefore\ A^*_{f,\varphi}\Gamma_\gamma &= K(\cdot, \varphi(\gamma(T))) - K(\cdot, \varphi(\gamma(0))).
\end{aligned}$$

□

The proof of THEOREM 2.7 generalizes the result of [36, Lemma 2] and [38, Proposition 5].

2.2.4 Null-Space

Theorem 2.8. *Let two holomorphic functions on a domain $\Omega \subseteq \mathbb{C}^n$ coincide on a nonempty open subset $U \subseteq \Omega$. Then, they are the same on the whole of Ω.*

Proof. Follow [22, Proposition 1.7.10] or [44, §1.2.2]. □

Theorem 2.9. *Let H be a reproducing kernel Hilbert space over the $\mathbb{C}$. Consider holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ and $\varphi : \mathbb{C} \to \mathbb{C}$ where $f \not\equiv 0$ and φ is non-constant. Let $A_{f,\varphi}$ be densely defined over H induced by two holomorphic functions f and φ. Then the null-space of $A_{f,\varphi}$*

is:

$$\text{(2.4)} \qquad \operatorname{nullsp}(A_{f,\varphi}) = \{g \in \mathbf{D}(A_{f,\varphi}) : g \textit{ is a constant function} \in \mathbb{C}\}.$$

Proof. Let $g \in \operatorname{nullsp}(A_{f,\varphi})$ and therefore, $A_{f,\varphi}g(z) \equiv g'(\varphi(z))\varphi'(z)f(z) = 0$ for all $z \in \mathbb{C}$ where $f \neq 0$ and φ is non-constant. Consider $\{w_m\}_{m \in \mathbb{Z}_+} \to w$ for some $w \in \mathbb{C}$ and $w_m \neq w_n$ if $m \neq n$. Holomorphicity of φ in $\mathbb{C}$ ensures $\varphi(w_m) \to \varphi(w)$. For the sake of contradiction, suppose there exist a sub-sequence $\{w_{mj}\}_j$ of $\{w_m\}_m$ that are the zeros of φ. Then, $\varphi(w) = 0$ and zeros of φ does accumulate. Therefore, $\varphi \equiv 0$ but this cannot happen as φ is non-constant. For sufficiently large m, $\varphi(w_m) \neq 0$.

Similarly $\varphi(w_m)$ has no constant sub-sequence. Thus, the selection of distinct $\varphi(w_m)$ is possible. Set $v_m = \varphi(w_m)$ and observe that $v_m \to \varphi(w)$ such that $g'(\varphi(w_m)) = g'(v_m) = 0$ for all $m \in \mathbb{Z}_+$. From above, we concluded that each one of $\{v_m\}_m$'s are distinct and $v_m \to \varphi(w)$ that they must have an accumulation point. Hence, by THEOREM 2.8, $g' \equiv 0$, yielding g a constant function in $\mathbb{C}$. $\square$

The following example demonstrates that we cannot provide the null-space for $A_{f,\varphi}$ over $\mathbb{C}^n$ unless $n = 1$.

Example 2.10. *Let $z = (z_1, z_2) \in \mathbb{C}^2$. Let $g : \mathbb{C}^2 \to \mathbb{C}$ defined by $g(z) = \nu_1 z_1 - \nu_2 z_2$ for some non-zero complex constants ν_1 and ν_2. Let $\varphi(z) = z$ and $f(z) = (\nu_1, \nu_2)^\top$. Then*

$$A_{f,\varphi}g(z) = \nabla g(\varphi(z))\, D\varphi(z) f(z) = \begin{bmatrix} \nu_2 & -\nu_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nu_1 & \nu_2 \end{bmatrix}^\top = \nu_2\nu_1 - \nu_1\nu_2 = 0.$$

Note that in above, the Euclidean dot product of $(\nabla g(\varphi(z)))^\top$ (which is $(\not\equiv 0)$) with that of $D\varphi(z)f(z)$ is 0 implying that the vectors $(\nabla g(\varphi(z)))^\top$ and $D\varphi(z)f(z)$ are mutually orthogonal in $\mathbb{C}^2$. Due to the Euclidean dot product being 0 here, it might tempt to include a linear function in several variables in the null-space of $A_{f,\varphi}$ over $\mathbb{C}^2$. However, vector valued *functions of several variables are not integral domains with respect to the Euclidean dot product.*

2.3 Boundedness

Similar to methods for weighted composition operators (cf. [4, 26]), determining a characterization of the symbols that result in a bounded Liouville weighted composition operator will rely on the action of the adjoint of these operators on kernel functions, which is established in LEMMA 2.11 over $\mathbf{F}^2(\mathbb{C}^n)$.

Lemma 2.11. *Suppose $f : \mathbb{C}^n \to \mathbb{C}^n$ and $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ are entire functions that yield a bounded $A_{f,\varphi}$ over $\mathbf{F}^2(\mathbb{C}^n)$, then*

$$\|A^*_{f,\varphi}K_{z_0}\|^2 = \left(f(z_0)^\top D\varphi(z_0)^\top (I_n + \varphi(z_0)\overline{\varphi(z_0)}^\top)\overline{D\varphi(z_0)f(z_0)}\right) e^{|\varphi(z_0)|^2}. \tag{2.5}$$

The quantity I_n in (2.5) *corresponds to the $n \times n$ identity matrix.*

Proof. By the definition of $A_{f,\varphi}$, following expression can be written for $z_0 \in \mathbb{C}^n$ with the help of reproducing kernel K_{z_0} at $z_0 \in \mathbb{C}^n$,

$$A_{f,\varphi}g(z_0) = \langle A_{f,\varphi}g, K_{z_0}\rangle = \langle g, A^*_{f,\varphi}K_{z_0}\rangle = \nabla g(\varphi(z_0))D\varphi(z_0)f(z_0).$$

For $z \in \mathbb{C}^n$, the evaluation of $A^*_{f,\varphi}K_{z_0}(z)$ may be determined through the inner product of the Hilbert space as

$$\begin{aligned} A^*_{f,\varphi}K_{z_0}(z) = \langle A^*_{f,\varphi}K_{z_0}, K_z\rangle = \langle K_{z_0}, A_{f,\varphi}K_z\rangle = \overline{\langle A_{f,\varphi}K_z, K_{z_0}\rangle} &= \overline{A_{f,\varphi}K_z(z_0)} \\ &= zK_{\varphi(z_0)}(z)\overline{D\varphi(z_0)f(z_0)}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\|A^*_{f,\varphi}K_{z_0}\|^2 &= \langle A_{f,\varphi}A^*_{f,\varphi}K_{z_0}, K_{z_0}\rangle \\
&= A_{f,\varphi}A^*_{f,\varphi}K_{z_0}(z_0) \\
&= \left(f(z_0)^\top D\varphi(z_0)^\top (I_n + \varphi(z_0)\overline{\varphi(z_0)}^\top)\overline{D\varphi(z_0)f(z_0)}\right) e^{|\varphi(z_0)|^2}.
\end{aligned}$$

□

The proof of LEMMA 2.11 may be adapted to show that K_z is in the adjoint of $A_{f,\varphi}$ for each $z \in \mathbb{C}$. Analogous to the approach in [18] the following quantities are defined.

Definition 2.12. *Let $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ and $f : \mathbb{C}^n \to \mathbb{C}^n$ be entire mappings that yield a bounded $A_{f,\varphi}$. Then the following are defined as*

$$\mathbf{M}_z(f,\varphi) := \frac{\|A^*_{f,\varphi}K_z\|^2}{\|K_z\|^2}, \quad \text{where } z \in \mathbb{C}^n \text{ and,} \tag{2.6}$$

$$\mathfrak{M}(f,\varphi) := \sup_{z\in\mathbb{C}^n} \mathbf{M}_z(f,\varphi) = \sup_{z\in\mathbb{C}^n} \frac{\|A^*_{f,\varphi}K_z\|^2}{\|K_z\|^2}. \tag{2.7}$$

Proposition 2.13. *Let $W : \mathbb{C}^n \to \mathbb{C}^n$ and $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ be entire functions with $W(z)$ non-vanishing. If*

$$|W(z)|^2 e^{|\varphi(z)|^2 - |z|^2} < C$$

for all $z \in \mathbb{C}^n$ then φ is linear (affine) i.e. $\varphi(z) = \mathcal{A}z + B$ where $B \in \mathbb{C}^n$ with $\|\mathcal{A}\| \leq 1$.

Proof. We begin by taking the logarithm of both sides,

$$2\log|W(z)| + \left(|\varphi(z)|^2 - |z|^2\right) < \log(C).$$

Now we introduce $z = (r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})^\top$. Now, $|z|^2 = \sum_{\ell=1}^n |z_\ell|^2$ and $|\varphi(z)|^2 = \sum_{\ell=1}^n |\varphi_\ell(z)|^2$, where $\varphi_\ell : \mathbb{C}^n \to \mathbb{C}$ is a component function of φ. If we fix a $k \in \{1, \dots, n\}$ we have

$$
\begin{aligned}
\log(C) &> 2\log|W(z)| + \left(|\varphi(z)|^2 - |z|^2\right), \\
&\geq 2\log|W(z)| + \left(|\varphi_k(z)|^2 - |z|^2\right), \\
&= 2\log|W(z)| + \left(|\varphi_k(z)|^2 - \sum_{\ell=1}^n r_\ell^2\right).
\end{aligned} \tag{2.8}
$$

We integrate both sides over $\mathbb{T}^n$

$$
\frac{2}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\log|W(z)|\,d\theta + \frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|\varphi_k(z)|^2 d\theta - \sum_{\ell=1}^n r_\ell^2 < \log(C).
$$

Again, the above is written using multi-index notation. By analyticity of $W(z)$, Jensen's inequality in PROPOSITION 2.4, and monotonicity of log and THEOREM 2.5, we get

$$
\begin{aligned}
2\log(|W(0)|) &= 2\log\left(\left|\frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\frac{W(\xi)}{\xi}\,d\xi\right|\right), \\
&\leq 2\log\left(\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|W(z)|\,d\theta\right), \\
&\leq \frac{2}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\log|W(z)|\,d\theta.
\end{aligned}
$$

We have,

$$
2\log(|W(0)|) + \frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|\varphi_k(z)|^2 d\theta - \sum_{\ell=1}^n r_\ell^2 < \log(C).
$$

By an application of our LEMMA 2.3,

$$
2\log(|W(0)|) + \sum_{|j|=0}|a_j^k|^2 r_1^{2j_1}\cdot\ldots\cdot r_n^{2j_n} - \sum_{\ell=1}^n r_\ell^2 < \log(C),
$$

here we used a super-script of k to indicate this a decomposition of the k-th component of the function φ. Let e_i be the multi-index with a 1 in the i-th spot and zeros else where. The above rearranges to:

$$2\log(|W(0)|) + \sum_{|j|=2} |a_j^k|^2 r_1^{2j_1} \cdot \ldots \cdot r_n^{2j_n} + |a_0^k|^2 + \sum_{\ell=0}^{n} (|a_{e_\ell}^k|^2 - 1) r_\ell^2 < \log(C).$$

This inequality is true for all $r = (r_1, \ldots, r_n)^\top \in \mathbb{R}^n$. As the second term is unbounded if any $|a_j^k| > 0$ for $j \geq 2$, this is enough to imply up to a relabelling that $\varphi_k(z) = a_{k,1}z_1 + \cdots + a_{k,n}z_n + b_k$. Thus, $\varphi(z) = \mathcal{A}z + B$ where, $\mathcal{A} = (a_{k,j})_{k,j=1}^{n,n}$ and $B = (b_1, \ldots, b_n)^\top$. From (2.8), notice the following:

$$|z|^2 + \log(C) > |\varphi(z)|^2.$$

Thus, it follows that

$$\lim_{|z|\to\infty} \frac{|\varphi(z)|}{|z|} < 1, \tag{2.9}$$

which is same to say $\lim_{|z|\to\infty} \frac{|\mathcal{A}z+B|}{|z|} < 1$. Now, suppose that $|\mathcal{A}\zeta| > |\zeta|$ for some ζ whose norm is 1. Setting $z = t\zeta$ and $t > 0$ in (2.9) yields $\lim_{t\to\infty} \frac{\left|A\zeta + \frac{1}{t}B\right|}{|\zeta|} < 1$, which is a contradiction and therefore, $\|\mathcal{A}\| \leq 1$. □

Remark 2.14. *If $\mathcal{A}$ is an invertible $n \times n$ complex-valued matrix, then $\varphi = \mathcal{A}z + B$ is an injective entire self-mapping on $\mathbb{C}^n$.*

Remark 2.15. *For the remainder of the chapter, φ will be an affine self-map on $\mathbb{C}^n$; that is to say that $\varphi(z) = \mathcal{A}z + B$ is an injective self-mapping entire function on $\mathbb{C}^n$ and $\mathcal{A}$ is invertible. This convention follows that found in [4, 18, 51] and [17] for the* affine *structure of φ.*

Following proposition delivers an important result when $\varphi(z) = \mathcal{A}z + B$ with $0 < \|\mathcal{A}\| < 1$.

Proposition 2.16. *The quantity $\mathbf{M}_z(f, \varphi)$ present in (1.5) belongs to $L^1(\mathbb{C}^n, dA(z))$, when $\varphi(z) = \mathcal{A}z + B$ where $\|\mathcal{A}\| < 1$.*

Proof. We are given that $\varphi(z) = \mathcal{A}z + B$ and $\|\mathcal{A}\| < 1$. Introduce $W_{\mathcal{A}}(z) = f(z)^\top \mathcal{A}^\top$ and $V_{\mathcal{A}}(z) = f(z)^\top \mathcal{A}^\top(\mathcal{A}z + B)$. To prove that $\mathbf{M}_z(f, \varphi) \in L^1(\mathbb{C}^n, dA(z))$, proceed as follows:

$$\begin{aligned}
\int_{\mathbb{C}^n} \mathbf{M}_z(f,\varphi)dA(z) &= \int_{\mathbb{C}^n} \frac{\|A^*_{f,\varphi}K_z\|^2}{\|K_z\|^2} dA(z) \\
&= \int_{\mathbb{C}^n} \left(W_{\mathcal{A}}(z)\overline{W_{\mathcal{A}}(z)}^\top + V_{\mathcal{A}}(z)\overline{V_{\mathcal{A}}(z)}^\top \right) e^{|\mathcal{A}z+B|^2} e^{-|z|^2} dA(z) \quad \text{(use (2.5))} \\
&\leq e^{|B|^2} \int_{\mathbb{C}^n} \left[|W_{\mathcal{A}}(z)|^2 + |V_{\mathcal{A}}(z)|^2 \right] e^{|\mathcal{A}z|^2 - |z|^2} dA(z).
\end{aligned}$$

A standard multi-variable limit argument shows that $[|W_{\mathcal{A}}(z)|^2 + |V_{\mathcal{A}}(z)|^2]\, e^{|\mathcal{A}z|^2 - |z|^2} \to 0$ as $|z| \to \infty$ whenever $0 < \|\mathcal{A}\| < 1$. Therefore, the last integral present above converges and the integral evaluation of it over $\mathbb{C}^n$ is finite. That is, $\int_{\mathbb{C}^n} [|W_{\mathcal{A}}(z)|^2 + |V_{\mathcal{A}}(z)|^2]\, e^{|\mathcal{A}z|^2 - |z|^2} dA(z) < \infty$, forcing in conclusion to have:

$$\int_{\mathbb{C}^n} \mathbf{M}_z(f,\varphi)dA(z) \leq e^{|B|^2} \int_{\mathbb{C}^n} \left[|W_{\mathcal{A}}(z)|^2 + |V_{\mathcal{A}}(z)|^2 \right] e^{|\mathcal{A}z|^2 - |z|^2} dA(z) < \infty.$$

Thus, $\|\mathbf{M}_z(f, \varphi)\|_{L^1(\mathbb{C}^n, dA(z))} < \infty$. □

The above proposition will play an important role in determining the bound of $A_{f,\varphi}$ over $\mathbf{F}^2(\mathbb{C}^n)$ discussed as follows.

2.3.1 Boundedness

Consider following notation:

$$W(z) \equiv (W_1(z), \ldots, W_n(z)) := f(z)^\top D\varphi(z)^\top. \tag{2.10}$$

Theorem 2.17 (Boundedness of $A_{f,\varphi}$)**.** *Let $f = (f_1, f_2, \cdots, f_n)^\top$ be an entire functions with $f : \mathbb{C}^n \to \mathbb{C}^n$ such that f is not identically the zero vector in $\mathbb{C}^n$ and φ be an entire functions with $\varphi : \mathbb{C}^n \to \mathbb{C}^n$. Given $A_{f,\varphi} : \mathbf{D}(A_{f,\varphi}) \to \mathbf{F}^2(\mathbb{C}^n)$ is bounded then,*

1. $\mathfrak{M}(f, \varphi) < \infty$,

2. *$\varphi(z)$ is affine with $\|\mathcal{A}\| \leq 1$,*

3. *if $\mathcal{A}$ is invertible then $\{f_i\}_{i=1}^n$ belongs to $\mathbf{F}^2(\mathbb{C}^n)$.*

4. *Furthermore, if $0 < \|\mathcal{A}\| < 1$, then the bound of $A_{f,\varphi}$ is $\sqrt{\frac{\|\mathbf{M}_z(f,\mathcal{A}z+B)\|_{L^1(\mathbb{C}^n,dA(z))}}{\pi^n}}$ over $\mathbf{F}^2(\mathbb{C}^n)$.*

Proof. Suppose that $A_{f,\varphi} : \mathbf{D}(A_{f,\varphi}) \to \mathbf{F}^2(\mathbb{C}^n)$ is bounded. Leveraging $\|A_{f,\varphi}\| = \|A^*_{f,\varphi}\|$, it follows that $\|A^*_{f,\varphi}\|$ is also bounded. Set $C := \|A_{f,\varphi}\|$.

1. For part (1), we have following for $z \in \mathbb{C}^n$,

$$\mathfrak{M}(f,\varphi) = \sup_{z\in\mathbb{C}^n} \frac{\|A^*_{f,\varphi}K_z\|^2}{\|K_z\|^2} \leq \|A^*_{f,\varphi}\|^2 = \|A_{f,\varphi}\|^2 < C < \infty. \tag{2.11}$$

2. By considering the notation in (2.10) and following Proposition 2.13, it follows that $\varphi(z) = \mathcal{A}z + B$ and $\|\mathcal{A}\| \leq 1$.

3. Now we prove that with $A_{f,\varphi}$ being bounded with invertible $\mathcal{A}$, each component of f that is $\{f_i\} \in \mathbf{F}^2(\mathbb{C}^n)$. For this, let π_i be the projection function from $\mathbb{C}^n$ to the i-th component. Then with $D\varphi(z) = D(\mathcal{A}z + B) = \mathcal{A}$, one can have following:

$$A_{f,\varphi}\pi_i(z) = (0,\dots,\underbrace{1}_{i^{\text{th}}\text{ entry}},\dots,0)\mathcal{A}f = (0,\dots,1,\dots,0)\mathcal{A}\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n. \end{bmatrix}. \tag{2.12}$$

   Since $\mathcal{A}$ happens to be invertible, this implies that the rows of $\mathcal{A}$ are linearly independent. Let $\vec{a}_i$ denote the i^{th} row of $\mathcal{A}$ and note that

$$(0,\dots,1,\dots,0)\mathcal{A} = \vec{a}_i.$$

By the above we know that the dot product of $\vec{a}_i$ and f is in $\mathbf{F}^2(\mathbb{C}^n)$ for all $i \in \{1, \dots, n\}$. Hence, linear combinations of these vectors are in $\mathbf{F}^2(\mathbb{C}^n)$ as well, i.e.

$$\left(\sum_i c_i \vec{a}_i\right) \cdot f \in \mathbf{F}^2(\mathbb{C}^n).$$

Since $\mathcal{A}$ is invertible and the rows thus form a basis for $\mathbb{C}^n$ we can construct the standard basis from linear combinations of $\vec{a}_i$, hence each component of f must also be in $\mathbf{F}^2(\mathbb{C}^n)$.

4. Assume $\varphi(z) = \mathcal{A}z + B$ with $\|\mathcal{A}\| < 1$, then Proposition (2.16) yields that $\mathbf{M}_z(f, \mathcal{A}z + B) \in L^1(\mathbb{C}^n, dA(z))$ implying $\|\mathbf{M}_z(f, \mathcal{A}z + B)\|_{L^1(\mathbb{C}^n, dA(z))} < \infty$. Take $g \in \mathbf{F}^2(\mathbb{C}^n)$ and then proceed as follows:

$$\begin{aligned}
\|A_{f,\varphi} g\|^2 &= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |A_{f,\varphi} g(z)|^2 d\mathbf{A}_{\mathbb{G}}(z) \\
&= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |A_{f,\varphi} g(z)|^2 e^{-|z|^2} dA(z) \\
&= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |\langle A_{f,\varphi} g, K_z \rangle|^2 e^{-|z|^2} dA(z), && \text{(reproducing property of } K_z) \\
&= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |\langle g, A^*_{f,\varphi} K_z \rangle|^2 e^{-|z|^2} dA(z) \\
&\leq \frac{1}{\pi^n} \int_{\mathbb{C}^n} \|g\|^2 \|A^*_{f,\varphi} K_z\|^2 e^{-|z|^2} dA(z), && \text{(Cauchy-Schwarz on } |\langle g, A^*_{f,\varphi} K_z \rangle|) \\
&= \frac{\|g\|^2}{\pi^n} \int_{\mathbb{C}^n} \frac{\|A^*_{f,\varphi} K_z\|^2}{\|K_z\|^2} dA(z) \\
&= \frac{\|g\|^2}{\pi^n} \int_{\mathbb{C}^n} \mathbf{M}_z(f, \mathcal{A}z + B) dA(z), && \text{(use (1.5))} \\
&= \frac{\|g\|^2}{\pi^n} \|\mathbf{M}_z(f, \mathcal{A}z + B)\|_{L^1(\mathbb{C}^n, dA(z))} \\
&< \infty, && \text{(conclude from Proposition (2.16)).}
\end{aligned}$$

Hence, taking the square-root of above delivers that the bound of $\|A_{f,\varphi}\| \leq \sqrt{\frac{\|\mathbf{M}_z(f,\varphi)\|_{L^1(\mathbb{C}^n, dA(z))}}{\pi^n}}$ over $\mathbf{F}^2(\mathbb{C}^n)$ with $\varphi(z) = \mathcal{A}z + B$ and $\|A\| < 1$.

□

2.4 Essential norm & Compactness

2.4.1 Essential norm of A_f, φ

We recall important terminologies here for this section.

Definition 2.18. *Consider the power series expansion of $f(z)$ around origin as $f(z) = \sum_{k=0}^{\infty} \tilde{f}(k)z^k$, then we define an operator Ω_m as follows,*

$$\Omega_m f(z) = \sum_{|k|=m}^{\infty} \tilde{f}(k)z^k. \tag{2.13}$$

The operator Ω_m defined in (2.13) *is self-adjoint and idempotent.*

For $g \in \mathbf{F}^2(\mathbb{C}^n)$ with Ω_m acting on it, for $z \in \mathbb{C}^n$, we have

$$|\Omega_m g(z)|^2 = |\langle \Omega_m g, K_z \rangle|^2 \leq \|g\|^2 \|\Omega_m K_z\|^2 = \|g\|^2 (\Omega_m K_z(z)) = \|g\|^2 \left(\sum_{|k|=m}^{\infty} \frac{|z|^{2k}}{k!} \right). \tag{2.14}$$

We recall the definition of essential norm of an operator as follows.

Definition 2.19. *For two Banach spaces $\mathbb{X}_1$ and $\mathbb{X}_2$ we denote by $K(\mathbb{X}_1, \mathbb{X}_2)$ the set of all compact operators from $\mathbb{X}_1$ into $\mathbb{X}_2$. The essential norm of a bounded linear operator $\mathrm{A} : \mathbb{X}_1 \to \mathbb{X}_2$, denoted as $\|\mathrm{A}\|_{\mathrm{e}}$ is defined as*

$$\|\mathrm{A}\|_{\mathrm{e}} := \inf \{ \|\mathrm{A} - T\| : T \in K(\mathbb{X}_1, \mathbb{X}_2) \}.$$

On the account of the fact that $\|\mathrm{A}_{f,\varphi}\|_{\mathrm{e}} = 0$ if and only if A is compact.

We will use the following notion of *compactness* for $A_{f,\varphi}$ on $\mathbf{F}^2(\mathbb{C}^n)$.

Proposition 2.20. *A bounded and linear operator $\mathcal{T}$ is compact on $\mathbf{F}^2(\mathbb{C}^n)$ if and only if $\lim_{m\to\infty} \|\mathcal{T} g_m - \mathcal{T} g\| \to 0$ whenever $\{g_m\}_{m=0}^{\infty} \to g$ weakly.*

The following is the notion of weakly convergence in $\mathbf{F}^2(\mathbb{C}^n)$ that we adopt in this paper.

Proposition 2.21. *Let $\{g_m\}_{m=0}^{\infty}$ be any sequence in $\mathbf{F}^2(\mathbb{C}^n)$. Then the sequence $\{g_m\}_{m=0}^{\infty}$ converges to 0 weakly if and only if following are true,*

1. *the sequence $\{g_m\}_{m=0}^{\infty}$ is bounded in the norm in $\mathbf{F}^2(\mathbb{C}^n)$,*
2. *$\{g_m\}_{m=0}^{\infty}$ is uniformly convergent to 0 on the compact subsets of $\mathbb{C}^n$.*

The immediate consequence of PROPOSITION 2.21 is following.

Corollary 2.22. *Suppose we have sequence $\{w_m\}_{m=0}^{\infty} \in \mathbb{C}^n$ such that $|w_m| \to \infty$ as $m \to \infty$. The sequence of normalized kernel (from (1.3)) given as $g_m = k_{\varphi(w_m)}$, where $\varphi(w_m) = \mathcal{A}w_m + B$ with $\mathcal{A}$ being a complex $n \times n$ matrix with $B \in \mathbb{C}^n$, converges weakly to 0 in $\mathbf{F}^2(\mathbb{C}^n)$.*

Lemma 2.23. *Consider $f : \mathbb{C}^n \to \mathbb{C}^n$ and $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ as an entire mappings. Assuming that $A_{f,\varphi} : \mathbf{D}(A_{f,\varphi}) \to \mathbf{F}^2(\mathbb{C}^n)$ is bounded, then the essential norm of $A_{f,\varphi}$ enjoys the following inequality:*

$$\|A_{f,\varphi}\|_{\mathrm{e}} \leq \liminf_{m\to\infty} \|A_{f,\varphi}\Omega_m\|, \tag{2.15}$$

where Ω_m is the operator introduced in (2.13).

Proof. Let $\mathfrak{A}$ be any compact operator on $\mathbf{F}^2(\mathbb{C}^n)$, then

$$\|A_{f,\varphi} - \mathfrak{A}\| \leq \|A_{f,\varphi}\Omega_m\| + \|A_{f,\varphi}(I - \Omega_m) - \mathfrak{A}\|. \tag{2.16}$$

The operator I means the identity operator. Taking the infimum over compact operators $\mathfrak{A}$ and letting $m \to \infty$ in (2.16) we have following:

$$\begin{aligned}
\liminf_{m\to\infty} \|A_{f,\varphi} - \mathfrak{A}\| &\leq \liminf_{m\to\infty} \|A_{f,\varphi}\Omega_m\| + \liminf_{m\to\infty} \|A_{f,\varphi}(I - \Omega_m) - \mathfrak{A}\|, \\
&= \liminf_{m\to\infty} \|A_{f,\varphi}\Omega_m\| + 0, \\
&= \liminf_{m\to\infty} \|A_{f,\varphi}\Omega_m\|.
\end{aligned}$$

Since, $I - \Omega_m$ is compact on $\mathbf{F}^2(\mathbb{C}^n)$, so $\|A_{f,\varphi}(I - \Omega_m)\|_{\mathrm{e}} = 0$ and thus, $\liminf_{m\to\infty}\|A_{f,\varphi}(I - \Omega_m) - \mathfrak{A}\| = 0$. We used this fact from second step to third step to conclude the result. Thus, we establish (2.15). $\square$

2.4.1.1 Essential norm of $A_{f,\varphi}$

Theorem 2.24 (Bounds of $\|A_{f,\varphi}\|_{\mathrm{e}}$). *Consider $A_{f,\varphi} : \mathbf{D}(A_{f,\varphi}) \to \mathbf{F}^2(\mathbb{C}^n)$ to be bounded, induced by entire functions $f : \mathbb{C}^n \to \mathbb{C}^n$ with $f \not\equiv 0$, and φ to be affine with $\|\mathcal{A}\| < 1$. Then the essential norm estimates of $A_{f,\varphi}$ is given as follows:*

$$\limsup_{|z|\to\infty} \sqrt{\mathbf{M}_{\mathcal{A}z+B}(f, \mathcal{A}z + B)} \leq \|A_{f,\varphi}\|_{\mathrm{e}} \leq \pi^{-n/2}\sqrt{\|\mathbf{M}_z(f, \mathcal{A}z + B)\|_{L^1(\mathbb{C}^n, dA(z))}}.$$

Moreover, if $\mathcal{A}$ is invertible, then φ is an automorphsim of $\mathbb{C}^n$ and we have that

$$\|A_{f,\varphi}\|_{\mathrm{e}} = \limsup_{|z|\to\infty} \sqrt{\mathbf{M}_z(f, \mathcal{A}z + B)}.$$

Proof. Consider a compact operator F over $\mathbf{F}^2(\mathbb{C}^n)$; by Schauder's theorem, F^* is also compact over $\mathbf{F}^2(\mathbb{C}^n)$. Meanwhile consider $\{z_m\}$ as a sequence such that $|z|_m \to \infty$ and by Corollary 2.22, $k_{\varphi(z_m)}$ converges weakly to 0 as $m \to \infty$ which is why $F^* k_{\varphi(z_m)} \to 0$ as $m \to \infty$. Therefore, we have following:

$$\begin{aligned}
\|A_{f,\varphi} - F\| &= \|A^*_{f,\varphi} - F^*\|, \\
&\geq \limsup_{m\to\infty} \|(A^*_{f,\varphi} - F^*)k_{\varphi(z_m)}\|, \\
&\geq \limsup_{m\to\infty} \left(\|A^*_{f,\varphi} k_{\varphi(z_m)}\| - \|F^* k_{\varphi(z_m)}\| \right), \\
&= \limsup_{m\to\infty} \|A^*_{f,\varphi} k_{\varphi(z_m)}\|, \\
&= \limsup_{|z|\to\infty} \sqrt{\mathbf{M}_{\varphi(z)}(f, \mathcal{A}z + B)}.
\end{aligned}$$

By application of THEOREM 2.17, we have the affine structure of $\varphi(z) = \mathcal{A}z + B$ and therefore the lower bound becomes $\limsup_{|z|\to\infty} \sqrt{\mathbf{M}_{\mathcal{A}z+B}(f, \mathcal{A}z + B)}$. This establishes the lower bound of $\|A_{f,\varphi}\|_{\mathrm{e}}$.

Working on the upper bound of $\|A_{f,\varphi}\|_{\mathrm{e}}$, as follows. We will use LEMMA 2.23 and due to the affine structure of φ, we have $\varphi(z) = \mathcal{A}z + B$ along with $0 < \|\mathcal{A}\| < 1$.

Take $g \in \mathbf{F}^2(\mathbb{C}^n)$ and fix $R > 0$, then,

$$\begin{aligned}
\|A_{f,\varphi}\Omega_m g\|^2 &= \int_{\mathbb{C}^n} |A_{f,\varphi}\Omega_m g(z)|^2 d\mathbf{A}_{\mathbb{G}}(z), \\
&= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |A_{f,\varphi}\Omega_m g(z)|^2 \, e^{-|z|^2} dA(z), \\
&= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |\langle \Omega_m g, A^*_{f,\varphi} K_z\rangle|^2 e^{-|z|^2} dA(z), \\
&= \frac{1}{\pi^n} \left[\left(\int_{|z|\le R} + \int_{|z|>R} \right) |\langle \Omega_m g, A^*_{f,\varphi} K_z\rangle|^2 e^{-|z|^2} dA(z) \right], \\
&= I_1 + I_2.
\end{aligned}$$

We will consider both I_1 and I_2 simultaneously. For I_1, we have:

$$\begin{aligned}
I_1 &= \frac{1}{\pi^n} \int_{|z|\le R} |\langle \Omega_m g, A^*_{f,\varphi} K_z\rangle|^2 e^{-|z|^2} dA(z) \\
&\le \frac{1}{\pi^n} \|g\|^2 \int_{|z|\le R} \left(\sum_{|k|=m}^{\infty} \frac{|z|^{2k}}{k!} \right) \|A^*_{f,\varphi} K_z\|^2 e^{-|z|^2} dA(z), && \text{(use result in (2.14))} \\
&\le \|g\|^2 \frac{1}{\pi^n} \left(\sum_{|k|=m}^{\infty} \frac{R^{2k}}{k!} \right) \int_{|z|\le R} \mathbf{M}_z(f, \varphi) dA(z), && \text{(use } |z| \le R\text{).}
\end{aligned}$$

Note that, since $g \in \mathbf{F}^2(\mathbb{C}^n)$ by choice, thus $\|g\|$ is finite-valued and so is $\int_{|z|\le R} \mathbf{M}_z(f, \varphi) dA(z)$. The latter one is finite-valued because $A_{f,\varphi}$ is bounded over $\mathbf{F}^2(\mathbb{C}^n)$ forcing $\|\mathbf{M}_z(f, \varphi)\|_{L^1(\mathbb{C}^n, dA(z))} < \infty$ by THEOREM 2.17 followed by the integral monotonicity of $\mathbf{M}_z(f, \varphi)$ on $\{z \in \mathbb{C}^n : |z| \le R\}$ vs. $z \in \mathbb{C}^n$. If $m \to \infty$, then $\left(\sum_{|k|=m}^{\infty} \frac{R^{2k}}{k!} \right) \to 0$ which implies that $\lim_{m\to\infty} I_1 = 0$.

For I_2, we have:

$$I_2 = \frac{1}{\pi^n} \int_{|z|>R} |\langle \Omega_m g, A^*_{f,\varphi} K_z\rangle|^2 e^{-|z|^2} dA(z)$$

$$
\begin{aligned}
&\leq \frac{1}{\pi^n}\int_{|z|>R}\|\Omega_m g\|^2\|A^*_{f,\varphi}K_z\|^2 e^{-|z|^2}dA(z), \qquad \text{(Cauchy-Schwarz on } |\langle\Omega_m g, A^*_{f,\varphi}K_z\rangle|)\\
&= \frac{\|\Omega_m g\|^2}{\pi^n}\int_{|z|>R}\mathbf{M}_z(f,\varphi)dA(z)\\
&\leq \frac{\|g\|^2}{\pi^n}\int_{|z|>R}\mathbf{M}_z(f,\varphi)dA(z), \qquad (\|\Omega_m g\|\leq \|g\|)\\
&\leq \frac{\|g\|^2}{\pi^n}\sup_R\left(\int_{|z|>R}\mathbf{M}_z(f,\varphi)dA(z)\right).
\end{aligned}
$$

Due to the integral monotonicity, $\int_{|z|>R}\mathbf{M}_z(f,\varphi)dA(z)\leq \|\mathbf{M}_z(f,\varphi)\|_{L^1(\mathbb{C}^n,dA(z))}<\infty$, therefore taking the 'sup' in the previous chain of inequalities results that $\sup_R\left(\int_{|z|>R}\mathbf{M}_z(f,\varphi)dA(z)\right)<\infty$. Now, recall the result of Lemma 2.23 as follows:

$$
\begin{aligned}
\|A_{f,\varphi}\|_{\mathrm{e}} &\leq \liminf_{m\to\infty}\|A_{f,\varphi}\Omega_m g\|\\
&\leq \sqrt{\frac{\|g\|^2}{\pi^n}\sup_R\left(\int_{|z|>R}\mathbf{M}_z(f,\varphi)dA(z)\right)}\\
&= \|g\|\sqrt{\frac{1}{\pi^n}}\sup_R\sqrt{\int_{|z|>R}\mathbf{M}_z(f,\varphi)dA(z)}.
\end{aligned}
$$

Since $\limsup_{R\to\infty}\sqrt{\int_{|z|>R}\mathbf{M}_z(f,\varphi)dA(z)}=\|\mathbf{M}_z(f,\varphi)\|_{L^1(\mathbb{C}^n,dA(z))}$, we get the desired conclusion after we let $R\to\infty$ in the above argument as explained. We now need only note that if φ is an automorphism of $\mathbb{C}^n$ that

$$
\limsup_{|z|\to\infty}\sqrt{\mathbf{M}_{\mathcal{A}z+B}(f,\mathcal{A}z+B)}=\limsup_{|z|\to\infty}\sqrt{\mathbf{M}_z(f,\mathcal{A}z+B)}.
$$

□

2.4.2 Compactness

Theorem 2.25 (Compactness of $A_{f,\varphi}$). *Consider the bounded $A_{f,\varphi}:\mathbf{D}(A_{f,\varphi})\to\mathbf{F}^2(\mathbb{C}^n)$ induced by an entire function $f\not\equiv 0$ and φ to be affine with $\|\mathcal{A}\|<1$. Then bounded $A_{f,\varphi}:\mathbf{D}(A_{f,\varphi})\to\mathbf{F}^2(\mathbb{C}^n)$ is compact if and only if* $\lim_{|z|\to\infty}\mathbf{M}_z(f,\mathcal{A}z+B)=0$.

Proof. We consider the bounded $A_{f,\varphi} : \mathbf{D}(A_{f,\varphi}) \to \mathbf{F}^2(\mathbb{C}^n)$ and we proceed the proof by first showing the $\implies$ direction by assuming that $A_{f,\varphi}$ is compact over $\mathbf{F}^2(\mathbb{C}^n)$. Since $A_{f,\varphi}$ is compact then by Schauder's theorem $A^*_{f,\varphi}$ is so over $\mathbf{F}^2(\mathbb{C}^n)$. In the light of the fact that $k_z = \frac{K_z}{\|K_z\|} \to 0$ weakly as $|z|\to \infty$ implies that $\frac{\|A^*_{f,\varphi}K_z\|^2}{\|K_z\|^2} \to 0$. This can be readily understand by realizing following chain of equalities:

$$\frac{\|A^*_{f,\varphi}K_z\|^2}{\|K_z\|^2} = \frac{\langle A^*_{f,\varphi}K_z, A^*_{f,\varphi}K_z\rangle}{\|K_z\|^2} = \left\langle A^*_{f,\varphi}\frac{K_z}{\|K_z\|}, A^*_{f,\varphi}\frac{K_z}{\|K_z\|}\right\rangle \to 0 \qquad \text{because} \qquad \frac{K_z}{\|K_z\|} \to 0.$$

Upon recalling (1.5) and THEOREM 2.17, we see that $\lim_{|z|\to\infty} \mathbf{M}_z(f, \mathcal{A}z + B) = 0$. For the converse, we note that if $\lim_{|z|\to\infty} \mathbf{M}_z(f, \mathcal{A}z + B) = 0$ then the essential norm is zero, hence the operator is compact. $\square$

The following example illustrates the boundedness, essential norm and compactness characterization of $A_{f,\varphi}$ over the $\mathbf{F}^2(\mathbb{C}^n)$ that is discussed in this paper. It also serves an example for [36, Corollary 1]. Following section provides an example for the bounded and compact $A_{f,\varphi}$.

2.5 Example

Example 2.26. *Suppose $z \in \mathbb{C}$. Let $f_N(z) = z^N$ where $N \in \mathbb{Z}_+$ and let $\varphi_{\mathcal{A}}(z) = \mathcal{A}z$ for some complex $\mathcal{A}$ such that $0 < |\mathcal{A}| < 1$. Incorporate* LEMMA *2.11 to yield:*

$$\mathbf{M}_z(f_N, \varphi_{\mathcal{A}}) = \frac{\|A^*_{f_N,\varphi_{\mathcal{A}}}K_z\|^2}{\|K_z\|^2} = |\mathcal{A}|^2\left(1 + |\mathcal{A}|^2|z|^2\right)|z|^{2N}\left[\exp\left(|\mathcal{A}|^2|z|^2 - |z|^2\right)\right].$$

With a direct calculation,

$$\mathfrak{M}(f_N, \varphi_{\mathcal{A}}) = \sup_{z\in\mathbb{C}}\left\{|\mathcal{A}|^2\left(1 + |\mathcal{A}|^2|z|^2\right)|z|^{2N}\left[\exp\left(|\mathcal{A}|^2|z|^2 - |z|^2\right)\right]\right\}, \tag{2.17}$$

and the value of $z \in \mathbb{C}$ *for which the supremum of argument present in* (2.17) *attains, is the solution*[1] *of* $p_4(|z|) = 0$, *where*

$$p_4(|z|) = |z|^4 + |z|^2 \left(\frac{N|\mathcal{A}|^2 + 2|\mathcal{A}|^2 - 1}{|\mathcal{A}|^4 - |\mathcal{A}|^2} \right) + \frac{N}{|\mathcal{A}|^4 - |\mathcal{A}|^2}.$$

Therefore

$$\mathfrak{M}(f_N, \varphi_{\mathcal{A}}) = \left\{ |\mathcal{A}|^2 \left(1 + |\mathcal{A}|^2 |z|^2\right) |z|^{2N} \left[\exp\left(|\mathcal{A}|^2|z|^2 - |z|^2\right)\right] : z \text{ satisfying } p_4(|z|) = 0 \right\} < \infty.$$

Now, we will determine the bound of $A_{f_N, \varphi_{\mathcal{A}}}$ *by showing that* $\mathbf{M}_z(f_N, \varphi_{\mathcal{A}}) \in L^1(\mathbb{C}, dA(z))$ *as follows:*

$$\|\mathbf{M}_z(f_N, \varphi_{\mathcal{A}})\|_{L^1(\mathbb{C}, dA(z))} = \int_{\mathbb{C}} |\mathcal{A}|^2 \left(1 + |\mathcal{A}|^2 |z|^2\right) |z|^{2N} \left[\exp\left(|\mathcal{A}|^2 |z|^2 - |z|^2\right)\right] dA(z).$$

Proposition 2.16 dictates that $W_{\mathcal{A}}(z) = |\mathcal{A}|^2 |z|^{2N}$ *and* $V_{\mathcal{A}}(z) = |\mathcal{A}|^4 |z|^{2N+2}$ *for this example. Observe that both* $W_{\mathcal{A}}(z) \left[\exp\left(-(1 - |\mathcal{A}|^2)|z|^2\right)\right]$ *and* $V_{\mathcal{A}}(z) \left[\exp\left(-(1 - |\mathcal{A}|^2)|z|^2\right)\right]$ *are* L^1 *integrable on* $\mathbb{C}$ *with respect to the usual Lebesgue area measure* $dA(z)$ *on* $\mathbb{C}$. *This is explicitly demonstrated as follows:*

(2.18)

$$\begin{aligned} |\mathcal{A}|^2 \int_{\mathbb{C}} |z|^{2N} \left[\exp\left(-(1 - |\mathcal{A}|^2)|z|^2\right)\right] dA(z) &= 2\pi |\mathcal{A}|^2 \int_0^\infty r^{2N} \left[\exp\left(-(1 - |\mathcal{A}|^2) r^2\right)\right] r dr \\ &= \pi |\mathcal{A}|^2 \frac{\Gamma(N+1)}{(1 - |\mathcal{A}|^2)^{N+1}}; \end{aligned}$$

(2.19)

$$\begin{aligned} |\mathcal{A}|^4 \int_{\mathbb{C}} |z|^{2N+2} \left[\exp\left(-(1 - |\mathcal{A}|^2)|z|^2\right)\right] dA(z) &= 2\pi |\mathcal{A}|^4 \int_0^\infty r^{2N+2} \left[\exp\left(-(1 - |\mathcal{A}|^2) r^2\right)\right] r dr \\ &= \pi |\mathcal{A}|^4 \frac{\Gamma(N+2)}{(1 - |\mathcal{A}|^2)^{N+2}}. \end{aligned}$$

[1] Let $\mathfrak{p} = \left(\frac{N|\mathcal{A}|^2 + 2|\mathcal{A}|^2 - 1}{|\mathcal{A}|^4 - |\mathcal{A}|^2} \right)$ and $\mathfrak{r} = \frac{N}{|\mathcal{A}|^4 - |\mathcal{A}|^2}$ then the solutions of $p_4(|z|)$ are $\pm\sqrt{\frac{-\mathfrak{p} \pm \sqrt{\mathfrak{p}^2 - 4\mathfrak{r}}}{2}}$.

As $N < \infty$ *and* $0 < |\mathcal{A}| < 1$*, thus the result present above in two parts are finite. Hence the bound of* $\|A_{f_N,\varphi_{\mathcal{A}}}\|$ *here is* $\sqrt{|\mathcal{A}|^2 \frac{\Gamma(N+1)}{(1-|\mathcal{A}|^2)^{N+1}} + |\mathcal{A}|^4 \frac{\Gamma(N+2)}{(1-|\mathcal{A}|^2)^{N+2}}}$ *as explained in* THEOREM *2.17. Also,* $f_N(z) \in \mathbf{F}^2(\mathbb{C})$ *as* $\|f_N\| = \sqrt{N!} < \infty$*. Since for the considered* $\varphi_{\mathcal{A}}(z)$*, its inverse is* $\varphi_{\mathcal{A}}^{-1}(z) = \frac{1}{\mathcal{A}} z$*, therefore* THEOREM *2.24 dictates to have*

$$\begin{aligned}
\|A_{f_N,\varphi_{\mathcal{A}}}\|_{\mathrm{e}} &= \limsup_{|z|\to\infty} \left[|\mathcal{A}|^2 \left(1 + |\mathcal{A}|^2|z|^2\right)|z|^{2N}\left[\exp\left(|\mathcal{A}|^2|z|^2 - |z|^2\right)\right]\right]^{\frac{1}{2}},\\
&= |\mathcal{A}| \limsup_{|z|\to\infty} \left[\left(1 + |\mathcal{A}|^2|z|^2\right)|z|^{2N}\left[\exp\left(|\mathcal{A}|^2 - 1\right)|z|^2\right]\right]^{\frac{1}{2}}.
\end{aligned}$$

As $0 < |\mathcal{A}| < 1$, $\limsup_{|z|\to\infty} \left[(1 + |\mathcal{A}|^2|z|^2)\,|z|^{2N}[\exp(|\mathcal{A}|^2 - 1)\,|z|^2]\right]^{\frac{1}{2}} = 0$ *and therefore the essential norm* $A_{f,\varphi}$ *is* 0*, hence it is compact over* $\mathbf{F}^2(\mathbb{C})$ *as established in* THEOREM *2.25.*

Had $|\mathcal{A}| > 1$ *been into account, then the evaluation in* (2.18) *and* (2.19) *will consequently not be finite, leading to* $\|\mathbf{M}_z(f_N, \varphi_{\mathcal{A}})\|_{L^1(\mathbb{C}, dA(z))} \not< \infty$*. As a result,* $A_{f_N,\varphi_{\mathcal{A}}}$ *is* ***not*** *bounded, let alone* $A_{f_N,\varphi_{\mathcal{A}}}$ *to be compact.*

2.6 Conclusion

In this chapter, we established the bound of $A_{f,\varphi}$ and its compactness as well, followed by the essential norm estimates over $\mathbf{F}^2(\mathbb{C}^n)$. A concrete example is also delivered to support the aforementioned claims. These results eventually shows and establishes the norm convergence of the DMD methods, which is superior to the *strong operator topology* convergence methods of the Koopman operators. The norm convergence is actually the *provable convergence guarantee* method that helps the practitioners in performing the data-driven methods, for instance in [36].

CHAPTER 3

WEIGHTED COMPOSITION OPERATORS OVER THE PALEY-WIENER SPACE

Detailed discussion for the interaction of *weighted composition operators* $W_{\psi,\varphi}$, with crucial topics for $\mathcal{EFET}$, in particular, *Phragmén-Lindelöf indicator function* and *the Pólya Representation theorem* in scope of PW_π^2 are made in this chapter.

The notion of Phragmén-Lindelöf indicator function for $\mathcal{EFET}$ is center to it and therefore, we recall the definition of it. The Phragmén-Lindelöf indicator function is defined as follows:

Definition 3.1. *With $z = re^{i\theta}$, adopt $|z|= r$ and $\textsc{am}\, z = \theta$. The function $h_f\left(\textsc{am}\, z\right)$ is called as the Phragmén-Lindelöf indicator function for an entire function f of exponential type:*

$$h_f\left(\textsc{am}\, z\right) := \limsup_{|z|\to\infty} \frac{\log|f(z)|}{|z|^\rho}$$

with respect to the order ρ. If τ represents the type of f then $\tau = \max h_f\left(\textsc{am}\, z\right)$ (cf. [41]). The Phragmén-Lindelöf indicator function for an $\mathcal{EFET}$ describes the growth of the function along the ray $\{z : \textsc{am}\, z = \vartheta\}$.

Recall the relevant introductory details from SUBSECTION 1.1.3. Then the analysis of $W_{\psi,\varphi}$ over PW_π^2 begun after giving the simple results given in PROBLEM 3.2.

Problem 3.2. *Let $f \in PW_\pi^2$. Then for all $0 \le x_1 < x < \infty$ on $\mathbb{R}$, prove following:*

$$\int_{x_1}^{x} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(t) \frac{e^{itx} - e^{itx_1}}{it} dt,$$

where $\|f\|^2_{L^2(\mathbb{R})} = \|\hat{f}\|^2_{L^2(-\pi,\pi)}$. *Furthermore, the following inequality is also true for* $f \in L^1((x_1,x))$

$$\left|\int_{x_1}^{x} f(t)dt\right| \leq \frac{\|f\|}{\sqrt{\pi}}\sqrt{(x_1+x)\left[\frac{4}{\pi}+2\,\mathrm{Si}\,\pi\right]}.$$

Proof. In order to establish the desired equality, observe that for all $f \in PW^2_\pi$, $f \in L^1((x_1,x))$ automatically[1] true. So proceed as follows:

$$\int_{x_1}^{x} f(x)dx = \int_{x_1}^{x}\int_{-\pi}^{\pi} \hat{f}(t)e^{itx}dtdx = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\hat{f}(t)\left(\int_{x_1}^{x} e^{itx}dx\right)dt = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\hat{f}(t)\frac{e^{itx}-e^{itx_1}}{it}dt.$$

Furthermore,

$$\begin{aligned}
\left|\int_{x_1}^{x} f(x)dx\right|^2 &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\hat{f}(t)\frac{e^{itx}-e^{itx_1}}{it}\right|^2 dt \\
&= \frac{1}{2\pi}\|\hat{f}\|^2_{L^2(\pi,\pi)}\int_{-\pi}^{\pi}\frac{|e^{itx}-e^{itx_1}|^2}{t^2}dt \\
&= \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{R})}\int_{-\pi}^{\pi}\frac{(\cos tx-\cos tx_1)^2+(\sin tx-\sin tx_1)^2}{t^2}dt \\
&= \frac{1}{\pi}\|f\|^2_{L^2(\mathbb{R})}\int_{-\pi}^{\pi}\frac{1-\cos(tx+tx_1)}{t^2}dt.
\end{aligned}$$

Introduce $y = t(x+x_1)$ to the indefinite integral of $\dfrac{1-\cos(tx+tx_1)}{t^2}dt$ as follows:

$$\begin{aligned}
\int\frac{1-\cos(tx+tx_1)}{t^2}dt &= \int\frac{1-\cos y}{\left(\dfrac{y}{x_1+x}\right)^2}\frac{dy}{x_1+x} \\
&= (x_1+x)\left[\frac{1-\cos y}{y}+\int\frac{\sin y}{y}dy\right] \\
\implies \int_{-\pi}^{\pi}\frac{1-\cos(tx+tx_1)}{t^2}dt &= (x_1+x)\left[\left(\frac{1-\cos y}{y}\right)\Big|_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\frac{\sin y}{y}dy\right]
\end{aligned}$$

[1] For $x_1 < x < \infty$ on $\mathbb{R}$, it can be observed as follows:

$$\left|\int_{x_1}^{x} f(x)dx\right|^2 \leq \left(\int_{x_1}^{x}|f(x)|dx\right)^2 \leq \int_{x_1}^{x}dx\int_{x_1}^{x}|f(x)|^2dx \leq (x-x_1)\int_{\mathbb{R}}|f(x)|^2dx = (x-x_1)\|f\|^2_{L^2(\mathbb{R})} < \infty.$$

Hence $f \in L^1((x_1,x))$.

$$= (x_1 + x)\left[\frac{4}{\pi} + 2\,\mathrm{Si}\,\pi\right].$$

□

Besides the interesting and simple result in PROBLEM 3.2, it helps in the application of $W_{\psi,\varphi}$ after we have established the compactness of it on PW_π^2.

3.1 Key Lemma

Following theorem is an application of Hadamard's factorization theorem.

Theorem 3.3. *(cf. [46, Exercise-12, Page-155]) A non-vanishing entire function of finite order along with its higher-order derivatives also non-vanishing is of the form of e^{Az+B} for some constants A and B.*

THEOREM 3.3 will play an essential role in establishing the key characterization of $W_{\psi,\varphi}$'s symbols over PW_π^2.

Proposition 3.4. *The relation* $\operatorname{sinc}(2|\zeta|) \le \operatorname{sinc}(\zeta - \overline{\zeta})$ *holds for any* $\zeta \in \mathbb{C}$.

Proof. Note that $(\zeta - \overline{\zeta}) = 2i\,\mathrm{Im}\,\zeta$ and $\operatorname{sinc}\zeta = \prod_{n=1}^{\infty}\left(1 - \frac{\zeta^2}{n^2}\right)$. Combine these results and proceed as follows:

$$\begin{aligned}
\operatorname{sinc}\left(\zeta - \overline{\zeta}\right) &= \prod_{n=1}^{\infty}\left(1 - \frac{(2i\,\mathrm{Im}\,\zeta)^2}{n^2}\right)\\
&= \prod_{n=1}^{\infty}\left(1 + \frac{(2\,\mathrm{Im}\,\zeta)^2}{n^2}\right)\\
&\ge \prod_{n=1}^{\infty}\left(1 - \frac{(2\,\mathrm{Im}\,\zeta)^2}{n^2}\right)\\
&\ge \prod_{n=1}^{\infty}\left(1 - \frac{(2|\zeta|)^2}{n^2}\right)\\
&= \operatorname{sinc}\left(2|\zeta|\right).
\end{aligned}$$

Thus the desired manipulation is established[2]. $\square$

Lemma 3.5. *Let $\psi : \mathbb{C} \to \mathbb{C}$ and $\varphi : \mathbb{C} \to \mathbb{C}$ be entire functions over $\mathbb{C}$. Suppose there exist a positive constant C such that $|\psi(z)|^2 \dfrac{\operatorname{sinc}\left(\varphi(z) - \overline{\varphi(z)}\right)}{\operatorname{sinc}(z - \overline{z})} < C$, then $\varphi(z) = Az + B$ where A and B are complex constant.*

Proof. If for some positive constant C, the relation $|\psi(z)|^2 \dfrac{\operatorname{sinc}\left(\varphi(z) - \overline{\varphi(z)}\right)}{\operatorname{sinc}(z - \overline{z})} < C$ is true then it is evident that following holds:

$$\frac{\operatorname{sinc}\left(\varphi(z) - \overline{\varphi(z)}\right)}{\operatorname{sinc}(z - \overline{z})} < \frac{C}{|\psi(z)|^2}.$$

It is evident by PROPOSITION 3.4 that $\operatorname{sinc}\left(\varphi(z) - \overline{\varphi(z)}\right) \geq \operatorname{sinc}(2|\varphi(z)|)$. Therefore,

$$\operatorname{sinc}(2|\varphi(z)|) < \frac{C}{|\psi(z)|^2} \operatorname{sinc}(z - \overline{z}) = \frac{C}{|\psi(z)|^2} \frac{\sin 2\pi i \operatorname{Im} z}{2\pi i \operatorname{Im} z} = \frac{C}{|\psi(z)|^2} \frac{\sinh(2\pi \operatorname{Im} z)}{2\pi \operatorname{Im} z}.$$

Use the definition on 'sinh' function, deduce that $\sinh(2\pi \operatorname{Im} z) < \left(\dfrac{e^{2\pi \operatorname{Im} z} + 1}{2}\right)$ and therefore the above inequality indicates that

$$\frac{\sin 2\pi|\varphi(z)|}{2\pi|\varphi(z)|} < \frac{C'}{|\psi(z)|^2} \frac{\left(e^{2\pi \operatorname{Im} z} + 1\right)}{2\pi r|\sin\theta|} \qquad (C' = C/2).$$

As $-\dfrac{1}{2\pi|\varphi(z)|} \leq \dfrac{\sin 2\pi|\varphi(z)|}{2\pi|\varphi(z)|}$ for any φ defined over $\mathbb{C}$, therefore following prevails:

$$|\varphi(z)| < \frac{|\psi(z)|^2 r|\sin\theta|}{C'\left(e^{2\pi \operatorname{Im} z} + 1\right)} < \frac{|\psi(z)|^2 r|\sin\theta|}{C'}.$$

Set $f(z) = e^{\varphi(z)}$ and observe following:

$$|f(z)| = |e^{\varphi(z)}| \leq \exp(|\varphi(z)|) \leq \exp\left(\frac{|\psi(z)|^2 \pi r|\sin\theta|}{C'}\right).$$

[2]Also noted by Pólya [31, Equation-(11) Page-68].

Now, we need the order of f. So denote the order of f as ρ. Compute the order of f as follows:

$$
\begin{aligned}
\rho &= \limsup_{r\to\infty} \frac{\log\log\left(\exp\left(\frac{|\psi(z)|^2\pi r|\sin\theta|}{C'}\right)\right)}{\log r} \\
&= \limsup_{r\to\infty} \frac{2\log|\psi(z)|+\log(r)+\log|\sin\theta|}{\log r} \\
\implies \int_0^{\tilde{\theta}} \rho\frac{d\theta}{2\pi} &= \int_0^{\tilde{\theta}} 1\frac{d\theta}{2\pi} + \int_0^{\tilde{\theta}} \limsup_{r\to\infty} \frac{2\log|\psi(re^{\theta})|}{\log r}\frac{d\theta}{2\pi} \\
\rho\frac{\tilde{\theta}}{2\pi} &= \frac{\tilde{\theta}}{2\pi} + \limsup_{r\to\infty} \int_0^{\tilde{\theta}} \frac{2\log|\psi(re^{\theta})|}{\log r}\frac{d\theta}{2\pi}.
\end{aligned}
$$

Approach $\tilde{\theta} \to 2\pi$ above to have following:

$$
\begin{aligned}
\lim_{\tilde{\theta}\to 2\pi} \rho\frac{\tilde{\theta}}{2\pi} &= \lim_{\tilde{\theta}\to 2\pi} \frac{\tilde{\theta}}{2\pi} + \lim_{\tilde{\theta}\to 2\pi} \limsup_{r\to\infty} \int_0^{\tilde{\theta}} \frac{2\log|\psi(re^{\theta})|}{\log r}\frac{d\theta}{2\pi} \\
\implies \rho &= 1 + 2\limsup_{r\to\infty} \frac{\log|\psi(0)|}{\log r} = 1.
\end{aligned}
$$

Hence ρ is 1 (finite). That implies that $e^{\varphi(z)}$ is of order $1 < \infty$. Note that neither f never vanishes nor its higher-order derivatives vanishes. Together with these facts on $f = e^{\varphi(z)}$, THEOREM 3.3 dictates that $\varphi(z) = Az + B$ for some constants A and B. □

3.2 Boundedness

Proposition 3.6. *Let $W_{\psi,\varphi} : \operatorname{dom}(W_{\psi,\varphi}) \to PW_\pi^2$ induced by entire functions $\psi \not\equiv 0$ and φ over $\mathbb{C}$. If there exists a finite positive constant M which bounds $\frac{\|W_{\psi,\varphi}^* K_z\|^2}{\|K_z\|^2}$, then $\varphi(z) = az + b$.*

Proof. Following should be acknowledged for $\frac{\|W_{\psi,\varphi}^* K_z\|^2}{\|K_z\|^2}$:

$$
\frac{\|W_{\psi,\varphi}^* K_z\|^2}{\|K_z\|^2} = |\psi(z)|^2 \frac{K_{\varphi(z)}(\varphi(z))}{K_z(z)} = |\psi(z)|^2 \frac{\operatorname{sinc}\left(\varphi(z) - \overline{\varphi(z)}\right)}{\operatorname{sinc}(z - \overline{z})}.
$$

Now, upon the application of LEMMA 3.5, the desired result is achieved. □

Theorem 3.7. *Let $W_{\psi,\varphi} : \operatorname{dom}(W_{\psi,\varphi}) \to PW_\pi^2$ be induced by entire functions $\psi \not\equiv 0$ and φ defined over $\mathbb{C}$. Then $W_{\psi,\varphi}$ acts bounded over the PW_π^2 if and only if $\psi \in PW_\pi^2$, $\varphi = az + b$, $0 < |a| \leq 1$ and*

$$\sup_{z\in\mathbb{C}} \left[|\psi(z)|^2 \frac{\operatorname{sinc}\left(\varphi(z) - \overline{\varphi(z)}\right)}{\operatorname{sinc}(z - \overline{z})} \right] < \infty.$$

Proof. Suppose $W_{\psi,\varphi}$ is bounded over PW_π^2, that is $\|W_{\psi,\varphi}^*\|^2_{PW_\pi^2} = \|W_{\psi,\varphi}\|^2_{PW_\pi^2} = M < \infty$. Then

$$|\psi(z)|^2 \frac{\operatorname{sinc}\left(\varphi(z) - \overline{\varphi(z)}\right)}{\operatorname{sinc}(z - \overline{z})} = \frac{|\psi(z)|^2 K_{\varphi(z)}(\varphi(z))}{K_z(z)} = \frac{\|W_{\psi,\varphi}^* K_z\|^2}{\|K_z\|^2} \leq \|W_{\psi,\varphi}^*\|^2 < \infty.$$

Take the supremum over $z \in \mathbb{C}$ above yields that $\sup_{z\in\mathbb{C}} \left[|\psi(z)|^2 \frac{\operatorname{sinc}\left(\varphi(z) - \overline{\varphi(z)}\right)}{\operatorname{sinc}(z - \overline{z})} \right] < \infty$, as desired. From above we observe that $\frac{\|W_{\psi,\varphi}^* K_z\|^2}{\|K_z\|^2} < M$, apply Proposition 3.6 to yield $\varphi(z) = az + b$. Suppose $a \in \mathbb{C}$ with $|a| > 1$. So, for $f \in PW_\pi^2$ with order ρ and type σ, $f \circ az$ is of the same order but type $|a|^\rho \sigma$. Given that $|a| > 1$, hence $\operatorname{dom}(W_{\psi,\varphi}) \not\subset PW_\pi^2$ from the previous conclusion. Thus, failing the map $W_{\psi,\varphi} : \operatorname{dom}(W_{\psi,\varphi}) \to PW_\pi^2$. Therefore $0 < |a| \leq 1$. Moreover, for $f(z) = \operatorname{sinc} z \in PW_\pi^2$ and hence $f(z) = \langle f(x), K_z(x) \rangle = \int_\mathbb{R} f(x) K_z(x) dx$ holds by the reproducing property. However, if $a = i$ then $f \circ az = \sinh z$ and $f \circ az = \frac{\sinh z}{z} \notin PW_\pi^2$ as

$$f \circ az = \frac{\sinh z}{z} = \int_\mathbb{R} \frac{\sinh x}{x} K_z(x) dx = \infty, \quad \text{as } \int_\mathbb{R} \left| \frac{\sinh x}{x} \right|^2 dx = \infty.$$

Due to the failure of reproducing property as demonstrated above while assuming that $a \in \mathbb{C}$, it leads to conclude that $a \notin \mathbb{C}$ but $a \in \mathbb{R}$. Now, proceed further by assuming that $\psi \in PW_\pi^2$ and $\varphi(z) = az + b$ where $b = b_1 + ib_2$. Pick $f \in PW_\pi^2$ and proceed as follows:

$$\begin{aligned} \|W_{\psi,\varphi} f\|^2_{PW_\pi^2} &= \int_\mathbb{R} |\psi(x)|^2 |f(\varphi(x))|^2 dx \\ &\leq \|\psi\|^2_{L^2(\mathbb{R})} \int_\mathbb{R} |f(ax + b_1 + ib_2)|^2 dx \end{aligned}$$

$$
\begin{aligned}
&= \|\psi\|^2_{L^2(\mathbb{R})}\int_{\mathbb{R}} |f(s+ib_2)|^2 \frac{ds}{|a|} \\
&\leq \frac{1}{|a|}\|\psi\|^2_{L^2(\mathbb{R})} e^{2\pi|b_2|}\|f\|^2_{L^2(\mathbb{R})}, \qquad\qquad \textit{(Plancherel-Pólya theorem)} \\
&< \infty.
\end{aligned}
$$

Hence the bound of $W_{\psi,\varphi}$ is $\sqrt{\frac{1}{|a|}e^{2\pi|\operatorname{Im} b|}}\|\psi\|_{L^2(\mathbb{R})}$ over PW^2_π. $\square$

3.2.1 Boundedness Application-1

Following theorem deals with the interaction of Phragmén-Lindelöf indicator function and PW^2_π.

Theorem 3.8. *Let $h(\text{AM}\,z)$ be the Phragmén-Lindelöf indicator function. Then:*

1. *Following inequality holds true:*

$$
h_f(\text{AM}\,z) \leq \limsup_{|z|\to\infty} \frac{\log\|K_z\|}{|z|^\rho} = \frac{1}{2}\limsup_{|z|\to\infty} \frac{\log\left(\sinh\left(2\pi|z|\sin \text{AM}\,z\right)\right)}{|z|^\rho},
$$

for all $f \in PW^2_\pi$ with respect to the order ρ.

2. *Let $W_{\psi,\varphi} : \operatorname{dom}(W_{\psi,\varphi}) \to PW^2_\pi$ be bounded over PW^2_π and $\varphi(z) = az + b$ where $0 < |a| \leq 1$ and $b = b_1 + ib_2$. Then*

$$
\begin{aligned}
h_{W_{\psi,\varphi}f}(\text{AM}\,z) &\leq \limsup_{|z|\to\infty} \frac{\log\|W^*_{\psi,\varphi}K_z\|}{|z|^\rho} \\
&= \limsup_{|z|\to\infty}\left[\frac{\log|\psi(z)|}{|z|^{\rho'}} + \frac{1}{2}\frac{\log\sinh\left(2\pi(a|z|\sin \text{AM}\,z + b_2)\right) - \log(a|z|\sin \text{AM}\,z + b_2)}{|z|^{\rho'}}\right],
\end{aligned}
$$

for all $f \in PW^2_\pi$ with respect to the order ρ' where $\rho' = \max\left(\text{order of } \psi, \text{order of } f\right)$.

Proof. 1. Let f be arbitrarily picked from PW^2_π with $\|f\|_{PW^2_\pi} < \infty$. Then $f(z) = \langle f, K_z\rangle$ holds by the reproducing property of K_z in PW^2_π. Proceed by the definition of Phragmén-

Lindelöf indicator function as follows:

$$\begin{aligned}
h_f(\text{AM } z) &= \limsup_{|z|\to\infty} \frac{\log|f(z)|}{|z|^\rho} \\
&\leq \limsup_{|z|\to\infty} \frac{\log|\|f\|\|K_z\||}{|z|^\rho} \\
&= \limsup_{|z|\to\infty} \frac{\log\|K_z\|}{|z|^\rho} \\
&= \frac{1}{2}\limsup_{|z|\to\infty} \frac{\log\left(\operatorname{sinc}\left(z-\overline{z}\right)\right)}{|z|^\rho} \\
&= \frac{1}{2}\limsup_{|z|\to\infty} \frac{\log\left(\sinh\left(2\pi|z|\sin \text{AM } z\right)\right)}{|z|^\rho}.
\end{aligned}$$

2. With $W_{\psi,\varphi} : \operatorname{dom}\left(W_{\psi,\varphi}\right) \to PW_\pi^2$, then $W_{\psi,\varphi}f(z) = \psi(z)f\left(\varphi(z)\right)$ for all $f \in PW_\pi^2$ with $\|f\|_{PW_\pi^2} < \infty$. Since $\|W_{\psi,\varphi}\|_{PW_\pi^2} < \infty$, thus $W_{\psi,\varphi}f(z) = \langle W_{\psi,\varphi}f, K_z\rangle$ via (again) the reproducing property of K_z in PW_π^2. Let $\rho' = \max\left(\text{order of } \psi, \text{order of } f\right)$. As $W_{\psi,\varphi}$ is bounded over PW_π^2, therefore $\varphi(z) = az + b_1 + ib_2$. Proceed (again) by the definition of Phragmén-Lindelöf indicator function as follows:

$$\begin{aligned}
&h_{W_{\psi,\varphi}f}\left(\text{AM } z\right) \\
&= \limsup_{|z|\to\infty} \frac{\log|W_{\psi,\varphi}f(z)|}{|z|^{\rho'}} \\
&= \limsup_{|z|\to\infty} \frac{\log|\langle W_{\psi,\varphi}f, K_z\rangle|}{|z|^{\rho'}} \\
&= \limsup_{|z|\to\infty} \frac{\log|\langle f, W_{\psi,\varphi}^* K_z\rangle|}{|z|^{\rho'}} \\
&\leq \limsup_{|z|\to\infty} \frac{\log\|f\| + \log\|W_{\psi,\varphi}^* K_z\|}{|z|^{\rho'}} \\
&= \limsup_{|z|\to\infty} \frac{\log\|W_{\psi,\varphi}^* K_z\|}{|z|^{\rho'}} \\
&= \limsup_{|z|\to\infty} \frac{\log\left(|\psi(z)|\sqrt{K_{\varphi(z)}(\varphi(z))}\right)}{|z|^{\rho'}} \\
&= \limsup_{|z|\to\infty} \frac{\log\left(|\psi(z)|\right)}{|z|^{\rho'}} + \frac{1}{2}\limsup_{|z|\to\infty} \frac{\log\sinh\left(2\pi \operatorname{Im}\varphi(z)\right) - \log \operatorname{Im}\varphi(z)}{|z|^{\rho'}}
\end{aligned}$$

$$= \limsup_{|z|\to\infty} \left[\frac{\log|\psi(z)|}{|z|^{\rho'}} + \frac{1}{2} \frac{\log \sinh\left(2\pi(a|z|\sin \text{AM}\, z + b_2)\right) - \log(a|z|\sin \text{AM}\, z + b_2)}{|z|^{\rho'}} \right].$$

Hence proved. □

3.2.2 Boundedness Application-2

Following is another interesting application of $W_{\psi,\varphi} : \text{dom}\,(W_{\psi,\varphi}) \to PW_\pi^2$ when $W_{\psi,\varphi}$ is bounded over the PW_π^2.

Theorem 3.9. *Let ϕ and $\hat{\phi}$ be two continuous function of period 2π whose respective Fourier coefficients c_n and $\hat{c}_n$ satisfy $\sum|nc_n|^2 < \infty$ and $\sum|n\hat{c}_n|^2 < \infty$. Let $W_{\psi,\varphi} : PW_\pi^2 \to PW_\pi^2$ be bounded and $\varphi(z) = az + b$, then following relation holds:*

$$\begin{aligned}\left[2\phi(\pi)W_{\psi,\varphi} - 2\hat{\phi}(\pi)\right] K(z,z) = &\frac{\psi\,(2i\,\text{Im}\,z)\,\varphi\,(2i\,\text{Im}\,z)}{\pi} \int_{-\pi}^{\pi} \left[\phi(t) - \phi(-t)dt\right] e^{i\varphi(2i\,\text{Im}\,z)t} dt \\ &- \frac{2i\,\text{Im}\,z}{\pi} \int_{-\pi}^{\pi} \left[\hat{\phi}(t) - \hat{\phi}(-t)\right] e^{-2t\,\text{Im}\,z} dt.\end{aligned}$$

Proof. Recall the following relationship for the $\mathcal{EFET}$, $f(z)$ that belongs to $L^2(\mathbb{R})$[3].

$$f(z) + 2\phi(\tau)\frac{\sin \tau z}{z} = z \int_{-\tau}^{\tau} \left[\phi(t) - \phi(-t)\right] e^{izt} dt, \tag{3.1}$$

where ϕ is continuous function with $\phi(\bullet + 2\tau) = \phi(\bullet)$ whose Fourier coefficients c_n satisfy $\sum|nc_n|^2 < \infty$. Let $W_{\psi,\varphi}$ be bounded over PW_π^2. Pick an arbitrary $f \in PW_\pi^2$, then following is evident from (3.1):

$$\psi(z) f\left(\varphi(z)\right) + 2\phi(\tau)\psi(z)\frac{\sin \tau\varphi(z)}{\varphi(z)} = \psi(z)\varphi(z) \int_{-\tau}^{\tau} \left[\phi(t) - \phi(-t)\right] e^{i\varphi(z)t} dt$$

$$W_{\psi,\varphi} f(z) + 2\phi(\tau)\psi(z)\frac{\sin \tau\varphi(z)}{\varphi(z)} = \psi(z)\varphi(z) \int_{-\tau}^{\tau} \left[\phi(t) - \phi(-t)\right] e^{i\varphi(z)t} dt. \tag{3.2}$$

[3] In fact, this is true for $L^p(\mathbb{R})$ setting as well. Follow [2, 23, 30] for this relationship held by the $\mathcal{EFET}$ in L^p setting.

According to the prerequisite, $W_{\psi,\varphi}$ is bounded over PW_π^2, thus by THEOREM 3.7, $\psi \in PW_\pi^2$ and hence $W_{\psi,\varphi}f(z) = \psi(z)f(\varphi(z))$ is of exponential type which belongs to $L^2(\mathbb{R})$. Hence following is true for $W_{\psi,\varphi}f(z)$

$$W_{\psi,\varphi}f(z) + 2\hat{\phi}(\tau)\frac{\sin \tau z}{z} = z\int_{-\tau}^{\tau}\left[\hat{\phi}(t) - \hat{\phi}(-t)\right]e^{izt}dt, \tag{3.3}$$

where (again) the Fourier coefficients of $\hat{\phi}$ as $\hat{c}_n$ satisfy $\sum |n\hat{c}_n|^2 < \infty$. Combine (3.2) and (3.3) to have following:

$$\begin{aligned}2\phi(\tau)\psi(z)\frac{\sin \tau\varphi(z)}{\varphi(z)} - 2\hat{\phi}(\tau)\frac{\sin \tau z}{z} =& \psi(z)\varphi(z)\int_{-\tau}^{\tau}\left[\phi(t) - \phi(-t)\right]e^{i\varphi(z)t}dt - \\ & z\int_{-\tau}^{\tau}\left[\hat{\phi}(t) - \hat{\phi}(-t)\right]e^{izt}dt.\end{aligned}$$

As both ϕ and $\hat{\phi}$ are continuous, therefore take the limit on τ to π to yield following:

$$\begin{aligned}2\phi(\pi)\psi(z)\frac{\sin \pi\varphi(z)}{\varphi(z)} - 2\hat{\phi}(\pi)\frac{\sin \pi z}{z} =& \psi(z)\varphi(z)\int_{-\pi}^{\pi}\left[\phi(t) - \phi(-t)\right]e^{i\varphi(z)t}dt - \\ & z\int_{-\pi}^{\pi}\left[\hat{\phi}(t) - \hat{\phi}(-t)\right]e^{izt}dt.\end{aligned}$$

Divide above equality by π to have following:

$$2\phi(\pi)\psi(z)\frac{\sin \pi\varphi(z)}{\pi\varphi(z)} - 2\hat{\phi}(\pi)\frac{\sin \pi z}{\pi z} = \frac{\psi(z)\varphi(z)}{\pi}\int_{-\pi}^{\pi}\left[\phi(t) - \phi(-t)\right]e^{i\varphi(z)t}dt - \frac{z}{\pi}\int_{-\pi}^{\pi}\left[\hat{\phi}(t) - \hat{\phi}(-t)\right]e^{izt}dt.$$

Replace z by $(z - \overline{z})$ above to have:

$$\begin{aligned}&2\phi(\pi)\psi(z-\overline{z})\frac{\sin \pi\varphi(z-\overline{z})}{\pi\varphi(z-\overline{z})} - 2\hat{\phi}(\pi)\frac{\sin \pi(z-\overline{z})}{\pi(z-\overline{z})} \\ =& \frac{\psi(z-\overline{z})\varphi(z-\overline{z})}{\pi}\int_{-\pi}^{\pi}\left[\phi(t) - \phi(-t)\right]e^{i\varphi(z-\overline{z})t}dt - \\ & \frac{(z-\overline{z})}{\pi}\int_{-\pi}^{\pi}\left[\hat{\phi}(t) - \hat{\phi}(-t)\right]e^{i(z-\overline{z})t}dt.\end{aligned}$$

The above setting now can be easily transferred to following:

$$
\begin{aligned}
2\phi(\pi)W_{\psi,\varphi}K(z,z) - 2\hat{\phi}(\pi)K(z,z) = & \frac{\psi(2i\,\mathrm{Im}\,z)\varphi(2i\,\mathrm{Im}\,z)}{\pi}\int_{-\pi}^{\pi}\left[\phi(t)-\phi(-t)\right]e^{i\varphi(2i\,\mathrm{Im}\,z)t}dt \\
& - \frac{(2i\,\mathrm{Im}\,z)}{\pi}\int_{-\pi}^{\pi}\left[\hat{\phi}(t)-\hat{\phi}(-t)\right]e^{i(2i\,\mathrm{Im}\,z)t}dt \\
\implies \left[2\phi(\pi)W_{\psi,\varphi} - 2\hat{\phi}(\pi)\right]K(z,z) = & \frac{\psi(2i\,\mathrm{Im}\,z)\varphi(2i\,\mathrm{Im}\,z)}{\pi}\int_{-\pi}^{\pi}\left[\phi(t)-\phi(-t)\right]e^{i\varphi(2i\,\mathrm{Im}\,z)t}dt \\
& - \frac{(2i\,\mathrm{Im}\,z)}{\pi}\int_{-\pi}^{\pi}\left[\hat{\phi}(t)-\hat{\phi}(-t)\right]e^{-(2\,\mathrm{Im}\,z)t}dt.
\end{aligned}
$$

This establishes the desired result. Hence proved. □

3.3 The Pólya Representation Theorem for $W_{\psi,\varphi}$ over PW_π^2

Following are the standard and obvious theory prevalent in the domain of $\mathcal{EFET}$ [27, Lecture-9]. The following build-up is essential for the smooth understanding of the upcoming result between $W_{\psi,\varphi}$ acting on PW_π^2 via the Laplace transform $\mathcal{L}$.

Definition 3.10. *A closed non-empty set E is convex in $\mathbb{C}$ if it contains the line segment joining any two points of the set E. Precisely, $tz_1 + (1-t)z_2 \in E,\ \forall\ z_1\ \&\ z_2 \in E$ holds.*

Let $0 \le \vartheta \le 2\pi$. Now, project E on the ray $\textsc{am}\,z = \vartheta$. Denote $k(\vartheta)$ the distance from the origin to the most remote point of this projection. That is the computation of,

$$
k(\vartheta) = \max_{z\in E}\left(\mathrm{Re}(ze^{-\vartheta})\right) = \max_{z\in E}\left(x\cos\vartheta + y\sin\vartheta\right) \tag{3.4}
$$

of this projection.

Definition 3.11. *The function $k(\vartheta)$ in* (3.4) *is called the* supporting function *of E.*

Indeed, points $z \notin E$ can easily be characterised by the fact that $x\cos\vartheta + y\sin\vartheta - k(\vartheta) > 0$.

Definition 3.12. *Let $H_0(\infty)$ be set of all functions that are holomorphic near ∞ and that vanish at ∞.*

Definition 3.13. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an $\mathcal{EFET}$ with type τ. The* Borel transform *of this f is given as*

$$\mathbf{F}(w) := \sum_{n=0}^{\infty} \frac{a_n n!}{w^n} \tag{3.5}$$

and lives in $H_0(\infty)$. The radius of convergence of $\mathbf{F}(w) = \sum_{n=0}^{\infty} \dfrac{a_n n!}{w^n}$ is in fact τ.

Definition 3.14. *The smallest convex compact set D containing all the singularities of $\mathbf{F}$ defined in* (3.5) *is called the* conjugate indicator diagram *of $f(z)$.*

We know that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an $\mathcal{EFET}$ if and only if its Borel transform $\mathbf{F}$ that are holomorphic near ∞ and $\mathbf{F}(\infty) = 0$ (cf. [27, 40]). That is:

Theorem 3.15 (The *Pólya* Representation Theorem)**.** *Let f be an $\mathcal{EFET}$ and Γ be a contour containing the conjugate indicator diagram of f then following relation holds*

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \mathbf{F}(w) e^{zw} dw. \tag{3.6}$$

The relation in (3.6) *is referred as the* Pólya *representation for f.*

The Borel transform deals with great importance in relation with Schrödinger equation and Schrödinger operator and provide important results in quantum mechanics [7, 14]. The function $\mathbf{F}$ present in (3.5) is the Laplace transform of f (that is $\mathcal{L}(f)$) under the justified strip of convergence. Additionally, it is also the analytic continuation of f. The justification for the Laplace transform of f is given as follows:

$$\mathcal{L}(f)[w] = \int_0^{\infty} \left(\sum_{n=0}^{\infty} a_n s^n \right) e^{-sw} ds = \sum_{n=0}^{\infty} a_n \int_0^{\infty} s^n e^{-sw} ds = \sum_{n=0}^{\infty} a_n n!\, w^{-n-1} = \mathbf{F}(w).$$

Term by term integration in $\int_0^{\infty} \sum_{n=0}^{\infty} |a_n| s^n e^{-xs} ds = \sum_{n=0}^{\infty} |a_n| n!\, x^{-n-1}$, makes sense by the applicability of Fubini's theorem. Here the RHS converges for $x > \tau$ since $\limsup_{n\to\infty} (n!\, a_n)^{\frac{1}{n}} = \tau$.

Theorem 3.16. *The Pólya Representation for $W_{\psi,\varphi}$ acting over PW_π^2 is given as follows:*

1. *Let ψ and φ be entire functions defined over $\mathbb{C}$. Let the Laplace transform of ψ be $\mathcal{L}(\psi)[w]$ with $\alpha_1 < \mathrm{Re}(w) < \beta_1$. Similarly, let the Laplace transform of φ^N be $\mathcal{L}(\varphi^N)[w]$ with $\alpha_2 < \mathrm{Re}(w) < \beta_2$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then the Laplace transform of $\psi \cdot f \circ \varphi$ is:*

$$\mathcal{L}(\psi \cdot f \circ \varphi)[w] = \frac{1}{2\pi i} \sum_{n=0}^{\infty} a_n \int_{c-i\infty}^{c+i\infty} \mathcal{L}(\psi)[s]\mathcal{L}(\varphi^n)[w-s]ds,$$

   where $\alpha_1 + \alpha_2 < \mathrm{Re}(w) < \beta_1 + \beta_2$ and $\alpha_1 < c < \beta_1$.

2. *Let $W_{\psi,\varphi} : \mathrm{dom}\,(W_{\psi,\varphi}) \to PW_\pi^2$ be bounded and $\varphi(z) = Az + B$ with $0 < |A| \leq 1$. Then following adaption for $W_{\psi,\varphi}$ acting on f holds true:*

$$W_{\psi,\varphi} f(z) = -\frac{1}{4\pi^2} \sum_{n=0}^{\infty} a_n \int_{\Gamma} \int_{c-i\infty}^{c+i\infty} \mathcal{L}(\psi)[s]\mathcal{L}(\varphi^n)[w-s]e^{zw}dsdw, \tag{3.7}$$

   for all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in PW_\pi^2$ where Γ is the contour containing the conjugate indicator diagram of $\mathcal{L}(W_{\psi,\varphi} f)$ $(= \mathcal{L}(\psi \cdot f \circ \varphi))$. The relation present in (3.7) between $W_{\psi,\varphi}$ and the Laplace transform of it symbols is a one-to-one relationship.

Proof. The proof are given as follows:

1. With the given formatting of f, it is self-explanatory that $f \circ \varphi(z) = \sum_{n=0}^{\infty} a_n \varphi^n(z)$ and hence $\psi(z) \cdot f \circ \varphi(z) = \psi(z) \cdot \sum_{n=0}^{\infty} a_n \varphi^n(z) = \sum_{n=0}^{\infty} a_n \psi(z) \cdot \varphi^n(z)$. Consider the Laplace transform of ψ and φ^n as given in the prerequisite along with the respective strip of convergence and then proceed as follows,

$$\begin{aligned}
\mathcal{L}(\psi \cdot f \circ \varphi)[w] &= \int_0^{\infty} \psi(s) \cdot f \circ \varphi(s) e^{-ws} ds \\
&= \sum_{n=0}^{\infty} a_n \int_0^{\infty} \psi(s) \cdot \varphi^n(s) e^{-ws} ds \\
&= \sum_{n=0}^{\infty} a_n \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{L}(\psi)[s]\mathcal{L}(\varphi^n)[w-s]ds \right),
\end{aligned}$$

where the last step is concluded from [3, Page-385] with additional details for the strip of convergence. Therefore, the desired result is achieved for the justified strip of convergence.

2. Since $W_{\psi,\varphi}$ is bounded over PW_π^2. Therefore, $\psi \in PW_\pi^2$ and affine structure of φ is already prevalent. As ψ is an $\mathcal{EFET}$, therefore its Borel transform exist which is eventually $\mathcal{L}(\psi)$. Moreover, the product of ψ with $f \circ (Az + B) \in PW_\pi^2$ will still be $\mathcal{EFET}$ not exceeding π and therefore, their Laplace transform also exists, which is eventually $\mathcal{L}(\psi \cdot f \circ \varphi)$, already derived in the preceding part. The relation in (3.7) is the one-to-one relationship between $W_{\psi,\varphi}$ and its symbols via $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We are now in position to apply the THEOREM 3.15 as follows:

$$\begin{aligned} W_{\psi,\varphi} f(z) &= \frac{1}{2\pi i} \int_\Gamma \mathcal{L}(W_{\psi,\varphi} f)[w] e^{zw} dw \\ &= \frac{1}{2\pi i} \int_\Gamma \mathcal{L}(\psi \cdot f \circ \varphi)[w] e^{zw} dw \\ &= -\frac{1}{4\pi^2} \sum_{n=0}^{\infty} a_n \int_\Gamma \int_{c-i\infty}^{c+i\infty} \mathcal{L}(\psi)[s] \mathcal{L}(\varphi^n)[w-s] e^{zw} ds dw. \end{aligned}$$

Hence, the desired result is achieved. □

3.4 Compactness

3.4.1 Compactness of $W_{\psi,\varphi}$ over PW_π^2

Let $N \in \mathbb{Z}_+$. With $\varphi_N(z) = \varphi \circ \varphi \circ \cdots \circ \varphi(z)$, i.e. N-times composition of φ by itself. Adopt following:

$$(W_{\psi,\varphi})^N f(z) = \psi(z)\psi(\varphi(z)) \cdots \psi(\varphi_{N-1}(z)) f(\varphi_N(z)). \tag{3.8}$$

Proposition 3.17. *Let $W_{\psi,\varphi} : \mathrm{dom}\,(W_{\psi,\varphi}) \to PW_\pi^2$ be bounded over PW_π^2 and $\psi \in PW_\pi^2$ and $\varphi(z) = az + b$ with $0 < |a| \leq 1$ and $b \in \mathbb{R}$. Consider $(W_{\psi,\varphi})^N$ as defined in (3.8) for*

$N \in \mathbb{Z}_+$. *Then* $(W_{\psi,\varphi})^N$ *is bounded over* PW_π^2 *and the bound is given as:* $\left\|(W_{\psi,\varphi})^N\right\|_{PW_\pi^2} \leq \left(\frac{\|\psi\|_{L^2(\mathbb{R})}}{\sqrt{|a|}}\right)^N$.

Proof. Let $W_{\psi,\varphi}$ be bounded over PW_π^2 and hence $\psi \in PW_\pi^2$ with $\varphi(z) = az + b$ where $0 < |a| \leq 1$. Easy calculation for N composition on $\varphi(z) = az + b$ yields $\varphi_N(z) = a^N z + b\sum_{l=0}^{N-1} a^l$. Given that $b \in \mathbb{R}$; then $\operatorname{Im}\varphi_N(x) = 0$. Pick an arbitrary $f \in PW_\pi^2$ then:

$$\left\|(W_{\psi,\varphi})^N f\right\|_{PW_\pi^2}^2 = \int_{\mathbb{R}} |\psi(x)|^2 |\psi(\varphi(x))|^2 \cdots |\psi(\varphi_{N-1}(x))|^2 |f(\varphi_N(x))|^2 dx.$$

Notice following for $\psi(\varphi_m(x))$ for $m = 1, \ldots, N-1$:

$$|\psi(\varphi_m(x))|^2 \leq \frac{\|\psi\|_{L^2(\mathbb{R})}^2}{2\pi} \int_{-\pi}^{\pi} \left|e^{it\varphi_m(x)}\right|^2 dt = \frac{\|\psi\|_{L^2(\mathbb{R})}^2}{2\pi} \int_{-\pi}^{\pi} e^{\operatorname{Im} t\varphi_m(x)} dt = \|\psi\|_{L^2(\mathbb{R})}^2. \tag{3.9}$$

For $f \in PW_\pi^2$, notice following for $f(\varphi_N(x))$,

$$\int_{\mathbb{R}} |f(\varphi_N(x))|^2 dx = \int_{\mathbb{R}} \left| f\left(a^N x + b\sum_{l=0}^{N-1} a^l\right)\right|^2 dx = \frac{\|f\|_{PW_\pi^2}^2}{|a|^N}. \tag{3.10}$$

Combine the results of (3.9) and (3.10) to deduce following:

$$\left\|(W_{\psi,\varphi})^N f\right\|_{PW_\pi^2}^2 \leq \|\psi\|_{L^2(\mathbb{R})}^2 \left(\|\psi\|_{L^2(\mathbb{R})}^2\right)^{N-1} \frac{\|f\|_{PW_\pi^2}^2}{|a|^N} = \frac{\left(\|\psi\|_{L^2(\mathbb{R})}\right)^{2N}}{|a|^N} \|f\|_{PW_\pi^2}^2.$$

Taking the square-root of above yields the desired result. □

Proposition 3.18. *Let* $W_{\psi,\varphi} : \operatorname{dom}(W_{\psi,\varphi}) \to PW_\pi^2$ *be bounded and* $\varphi(z) = az + b$ *where* $0 < |a| < 1$ *and* $b \in \mathbb{R}$. *Define* $\mathcal{B}_N = \left(\frac{\|\psi\|_{L^2(\mathbb{R})}}{\sqrt{|a|}}\right)^N$. *Then* $\mathcal{B}_N \to 0$ *as* $N \to \infty$ *if and only if* $\|\psi\|_{L^2(\mathbb{R})} < \sqrt{|a|}$.

Proof. Assume the prerequisite of the proposition. The proof is as follows:

$\Longrightarrow$ If $\|\psi\|_{L^2(\mathbb{R})}< \sqrt{|a|}$, then this imply that $\mathcal{B}_N < 1$ for all $N \in \mathbb{Z}_+$. Therefore, $\mathcal{B}_N \to 0$ assuming that $\|\psi\|_{L^2(\mathbb{R})}< \sqrt{|a|}$ is true.

$\Longleftarrow$ Suppose that $\mathcal{B}_N \to 0$ as $N \to \infty$, this imply that $\dfrac{\|\psi\|_{L^2(\mathbb{R})}}{\sqrt{|a|}} < 1$. Hence, $\|\psi\|_{L^2(\mathbb{R})}< \sqrt{|a|}$ given that $\mathcal{B}_N \to 0$ as $N \to \infty$.

Hence proved. $\square$

Following proposition dictates the weakly convergence of $\left\{(W_{\psi,\varphi})^{|n|} K_n\right\}_n$ over the PW^2_π.

Proposition 3.19. *Let $W_{\psi,\varphi} : \mathrm{dom}\,(W_{\psi,\varphi}) \to PW^2_\pi$ be bounded where $\varphi(z) = az + b$ with $0 < |a|< 1$ with $b \in \mathbb{R}$. Assume that $\|\psi\|^2_{PW^2_\pi}< |a|$. Consider $\left\{(W_{\psi,\varphi})^{|n|} K_n\right\}_n$ for $n \in \mathbb{Z}$ where $\{K_n\}_n$ is the orthonormal basis in PW^2_π. Then the sequence $\left\{(W_{\psi,\varphi})^{|n|} K_n\right\}_n$ converges weakly to 0 in PW^2_π.*

Proof. Conclude from Proposition 3.17 that $(W_{\psi,\varphi})^{|n|} K_n$ is bounded over PW^2_π and also it converges to 0 over $\mathbb{C}$ followed by the Proposition 3.18 under the mentioned conditions. Combining these arguments results into the weakly convergence of $\left\{(W_{\psi,\varphi})^{|n|} K_n\right\}_n$ over PW^2_π. $\square$

Hence, following is the *compactness* characterization of $W_{\psi,\varphi}$ over PW^2_π.

Corollary 3.20. *Let $W_{\psi,\varphi} : \mathrm{dom}\,(W_{\psi,\varphi}) \to PW^2_\pi$ be bounded with $\varphi(z) = az + b$ where $0 < |a|< 1$ and $b \in \mathbb{R}$. Then $W_{\psi,\varphi}$ acts compactly over the PW^2_π if and only if $\|\psi\|_{PW^2_\pi}< \sqrt{|a|}$.*

Proof. As following is true:

$$\left\|W_{\psi,\varphi}\left((W_{\psi,\varphi})^{|n|} K_n\right)\right\|^2_{PW^2_\pi} = \left\|\left((W_{\psi,\varphi})^{|n|+1} K_n\right)\right\|^2_{PW^2_\pi} \leq \mathcal{B}^2_{|n|+1}.$$

By Proposition 3.18, we already have $\mathcal{B}_{|n|+1} \to 0$ if and only if $\dfrac{\|\psi\|^2_{PW^2_\pi}}{|a|} < 1$. This shall imply that $\left\|W_{\psi,\varphi}\left((W_{\psi,\varphi})^{|n|} K_n\right)\right\|_{PW^2_\pi} \to 0$ as $n \to \pm\infty$ if and only if $\dfrac{\|\psi\|^2_{PW^2_\pi}}{|a|} < 1$. $\square$

3.4.2 Compactness Application

Following theorem further provides the existence of a Borel measure over $\mathbb{R}$ under the restrictions of $\operatorname{Im} z > 0$ for the $\operatorname{Im} \varphi$.

Theorem 3.21. *Let $W_{\psi,\varphi} : \operatorname{dom}(W_{\psi,\varphi}) \to PW_\pi^2$ be bounded. Suppose $0 < a \leq 1$ and $y > 0$ and also let $b_2 > 0$ where $b = b_1 + ib_2$. Consider $(W_{\psi,\varphi})^N$ acting over PW_π^2 with $\operatorname{Im} \varphi_N = a^N y + b_2 \sum_{j=0}^{N-1} a^j$. Then there exists a non-negative a Borel measure over $\mathbb{R}$, $\mu(t)$ such that*

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t^2 + y^2} = \frac{b_2 \pi}{y} \sum_{j=0}^{N-1} a^j.$$

In particular, letting $d\mu(t) = F(t)dt$ then $F(t)$ is essentially a positive constant function with $F(t) = b_2 \sum_{j=0}^{N-1} a^j$.

Proof. Observe that $\operatorname{Im} \varphi_N = a^N y + b_2 \sum_{j=0}^{N-1} a^j$ is non-negative and harmonic function for $\operatorname{Im} z > 0$. Thus following relationship holds due to the Poisson representation:

$$a^N y + b_2 \sum_{j=0}^{N-1} a^j = \rho y + \frac{y}{\pi} \int_{\mathbb{R}} \frac{d\mu(t)}{t^2 + y^2}$$

where $\rho = \lim_{y \to \infty} \dfrac{\operatorname{Im} \varphi_N}{y} = a^N > 0$. Putting this value of ρ, the above setup simplifies to the following:

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t^2 + y^2} = \frac{b_2 \pi}{y} \sum_{j=0}^{N-1} a^j.$$

Additionally $\int_{\mathbb{R}} \dfrac{d\mu(t)}{1 + t^2} < \infty$ where $\mu(t)$ is the desired Borel measure over $\mathbb{R}$. Since we have established this representation for $\operatorname{Im} \varphi_N$, then we can apply Fatou's Theorem to

$d\mu(t) = F(t)dt$ to yield

$$F(t) = \lim_{y \to 0} \operatorname{Im} \varphi_N = b_2 \sum_{j=0}^{N-1} a^j.$$

Also, this $F(t)$ is $L^1(\mathbb{R})$ integrable with respect to $\dfrac{1}{1+t^2}$ as can be seen as follows:

$$\int_{\mathbb{R}} \frac{|F(t)|}{1+t^2} dt = b_2 \sum_{j=0}^{N-1} a^j \int_{\mathbb{R}} \frac{1}{1+t^2} dt = \pi b_2 \sum_{j=0}^{N-1} a^j < \infty.$$

$\square$

Theorem 3.22. *Let $W_{\psi,\varphi} : \operatorname{dom}(W_{\psi,\varphi}) \to PW_\pi^2$ be compact, then following attributes are carried over ψ:*

1. *ψ is bounded over $\mathbb{R}$.*
2. *Let ψ be of $\mathcal{EFET}$ with type τ.*
   (a) *There exists $\tilde{\psi}_n$ which are of bounded variation on $(-\tau, \tau)$ for all n such that for each z, following holds:*

   $$\psi(z) = \lim_{n \to \infty} \int_{-\tau}^{\tau} e^{izt} d\tilde{\psi}_n(t).$$

   (b) *Following inequality is true with no possibility of improving the constants for ψ*

   $$\left| \psi\left(x + \frac{\pi}{2\tau}\right) + \psi\left(x - \frac{\pi}{2\tau}\right) \right| \leq \frac{8}{\tau} \sup_n \left| \psi\left(\frac{n\pi}{\tau}\right) \right|.$$

   (c) *Suppose ψ is real for real x and $h_\psi\left(\pm\frac{\pi}{2}\right) \leq c$ and $|\psi(x)| \leq M$ for all $x \in \mathbb{R}$ then*

   $$|\psi(x+iy)| \leq M \cosh cy,$$

and there is equality at some non-real point only if $\psi(z) = M\cos(cz + B)$ *where* $B \in \mathbb{R}$.

Proof. Following are the respective proof for each attributes on ψ.

1. Consider now that $W_{\psi,\varphi}$ is compact over PW_π^2, then by the COROLLARY 3.20

$$\|\psi\|_{PW_\pi^2}^2 = \int_{\mathbb{R}} |\psi(x)|^2 dx < 2\pi|a|e^{-\sigma|b_2|}.$$

Therefore ψ is bounded over $\mathbb{R}$ by [27, Remark-Page-51].

2. (a) This result follows from [2, Theorem-6.8.14] when combined with the above result of boundedness of ψ over $\mathbb{R}$.

   (b) This result follows from [2, Theorem-11.5.4] when combined with the above result of boundedness of ψ over $\mathbb{R}$.

   (c) This result follows from [9, Page-556] or [32, Theorem-4, Page-826] or [2, Theorem-3, Page-83].

□

Theorem 3.23. *Let* $W_{\psi,\varphi} : \mathrm{dom}(W_{\psi,\varphi}) \to PW_\pi^2$ *be compact and* $\varphi(z) = |a|z + b$ *where* $b \in \mathbb{R}_+$. *Consider* $I_{\mathfrak{x}} = [0, \mathfrak{x}] \subset \mathbb{R}$. *Let* $f \in PW_\pi^2$ *which is bounded over* $\mathbb{R}$. *Additionally suppose both* f *and* ψ *are monotonic on the real axis, then*

$$|W_{\psi,\varphi}f(x)| \leq \frac{\|\psi\|\|f\|}{\pi^3}\left[\frac{4}{\pi} + 2\,\mathrm{Si}\,\pi\right]\sup_{x\in I_{\mathfrak{x}}}\sqrt{x\varphi(x)} = \frac{\|\psi\|\|f\|}{\pi^3}\left[\frac{4}{\pi} + 2\,\mathrm{Si}\,\pi\right]\sup_{x\in I_{\mathfrak{x}}}\sqrt{|a|x^2 + bx}.$$

The equality above will holds when

$$\psi = \frac{\sin^2(\tau_\psi \eta/2)}{\eta^2}, \quad \textit{and}$$
$$f = \frac{\sin^2(\tau_f \eta/2)}{\eta^2},$$

almost everywhere (a.e.)[4] *where* τ_ψ *and* τ_f *are the respective exponential type of* ψ *and* f *in* PW_π^2.

Proof. Follow [2, Theorem 11.3.7, Page 212] along with the setting of the *Fundamental theorem of Calculus*. The result is now established as follows:

$$\begin{aligned}
|W_{\psi,\varphi}f(x)| &= |\psi(x)f(\varphi(x))| \\
&\leq \frac{1}{\pi^2}\sup_{x\in I_{\mathfrak{r}}}\left|\int_0^x \psi(t)dt\right| \cdot \sup_{x\in I_{\mathfrak{r}}}\left|\int_0^{\varphi(x)} f(t)dt\right| \\
&= \frac{1}{\pi^2}\frac{\|\psi\|}{\sqrt{\pi}}\sup_{x\in I_{\mathfrak{r}}}\left\{\sqrt{x\left[\frac{4}{\pi}+2\,\mathrm{Si}\,\pi\right]}\right\}\cdot\frac{\|f\|}{\sqrt{\pi}}\sup_{x\in I_{\mathfrak{r}}}\left\{\sqrt{\varphi(x)\left[\frac{4}{\pi}+2\,\mathrm{Si}\,\pi\right]}\right\} \\
&= \frac{\|\psi\|\|f\|}{\pi^3}\left[\frac{4}{\pi}+2\,\mathrm{Si}\,\pi\right]\sup_{x\in I_{\mathfrak{r}}}\sqrt{x\varphi(x)} \\
&= \frac{\|\psi\|\|f\|}{\pi^3}\left[\frac{4}{\pi}+2\,\mathrm{Si}\,\pi\right]\sup_{x\in I_{\mathfrak{r}}}\sqrt{|a|x^2+bx}.
\end{aligned}$$

In the perspective of achieving equality again [2, Theorem 11.3.7, Page 212] to realize that

$$\begin{aligned}
\int_0^z \psi(t)dt &= \int_0^z \frac{\sin^2(\tau_\psi\eta/2)}{\eta^2}d\eta \\
\int_0^z f(t)dt &= \int_0^z \frac{\sin^2(\tau_f\eta/2)}{\eta^2}d\eta
\end{aligned}$$

where τ_ψ and τ_f are the respective type of ψ and f and obviously $\tau_\psi\cdot\tau_f\leq\pi$. Therefore, following are the additional consequences of above equalities:

$$\begin{aligned}
\int_0^z \psi(t)dt - \int_0^z \frac{\sin^2(\tau_\psi\eta/2)}{\eta^2}d\eta = 0 &\implies \psi = \frac{\sin^2(\tau_\psi\eta/2)}{\eta^2} \quad a.e., \\
\int_0^z f(t)dt - \int_0^z \frac{\sin^2(\tau_f\eta/2)}{\eta^2}d\eta = 0 &\implies f = \frac{\sin^2(\tau_f\eta/2)}{\eta^2} \quad a.e.
\end{aligned}$$

□

[4] If $\int_X f = \int_X g$, then $f = g$, *a.e.* over X.

3.5 Hilbert-Schmidtness Property

Theorem 3.24. *Let* $W_{\psi,\varphi} : \operatorname{dom}(W_{\psi,\varphi}) \to PW_\pi^2$ *be compact with* $\varphi(z) = az + b$ *where* $0 < |a| < 1$ *and* $b \in \mathbb{R}$*. Then* $W_{\psi,\varphi}$ *is not Hilbert-Schmidt over the* PW_π^2*.*

Proof. From the preliminary, $\{K_n\}_{n\in\mathbb{Z}}$ are the orthonormal basis for the PW_π^2. Therefore,

$$\begin{aligned}
\|W_{\psi,\varphi}\|_{\mathrm{HS}}^2 &= \sum_{n\in\mathbb{Z}} \|W_{\psi,\varphi} K_n\|_{PW_\pi^2}^2 \\
&\le \|\psi\|_{L^2(\mathbb{R})}^2 \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \left| \frac{\sin(\pi(ax+b-n))}{\pi(ax+b-n)} \right|^2 dx \\
&= \frac{\|\psi\|_{L^2(\mathbb{R})}^2}{|a|} \sum_{n\in\mathbb{Z}} 1 = \infty.
\end{aligned}$$

Thus, $W_{\psi,\varphi}$ is not Hilbert-Schmidt over PW_π^2. □

CHAPTER 4

COMPOSITION OPERATORS OVER THE POLY-LOGARITHMIC HARDY SPACE

After providing all the basic details related to $PL^2(\mathbb{D}; s)$ in SUBSECTION 1.1.5, this chapter aims to provide further in-depth analysis on $PL^2(\mathbb{D}; s)$. The target interest is the determination of the Nevanlinna counting function for the $PL^2(\mathbb{D}; s)$ and its consequences related to the composition operator (or, Koopman C_ϕ). Now, we move to following section where we develop the crucial equation of *Littlewood-Paley Identity* for $PL^2(\mathbb{D}; s)$.

4.1 Littlewood-Paley Identity for the Poly-Logarithmic Hardy Space

Recall that $f_s \in PL^2(\mathbb{D}; s)$ is entire in s and analytic for $|z|< 1$. We will use f_s as being holomorphic to establish the Littlewood-Paley Identity for $PL^2\,(\mathbb{D}; s)$ in THEOREM 4.1. The theory developed in this section will be employed further for defining the Nevanlinna counting function for $PL^2(\mathbb{D}; s)$.

Theorem 4.1. *If f_s belongs to $PL^2(\mathbb{D}; s)$, then following equality is true:*

$$\|f_s\|^2_{PL^2(\mathbb{D};s)} = |f_s(0)|^2 + \frac{1}{\pi\Gamma(2-2s)} \int_{\mathbb{D}} |f_s'(z)|^2 \left(2\log\frac{1}{|z|}\right)^{1-2s} dA(z). \tag{4.1}$$

The equality (4.1) *is the Littlewood-Paley Identity for $PL^2(\mathbb{D}; s)$.*

Proof. The *LHS* of (4.1) is straightforward, that is $\sum_{k=1}^{\infty}|\hat{f}(k)|^2$, followed by the definition of $f_s \in PL^2(\mathbb{D}; s)$ given in (1.18). Now, by the virtue of the fact $f_s(0) = 0$, thus we focus on the remaining part of RHS of the (4.1) as follows,

$$\begin{aligned}
\int_{\mathbb{D}}|f_s'(z)|^2\left(2\log\frac{1}{|z|}\right)^{1-2s}dA(z) &= \int_{\mathbb{D}}\left|\sum_{k=1}^{\infty}\frac{k\hat{f}(k)}{k^s}z^{k-1}\right|^2\left(2\log\frac{1}{|z|}\right)^{1-2s}dA(z),\\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{kl}{k^s l^s}\hat{f}(k)\overline{\hat{f}(l)}2\pi\delta_{kl}\int_0^1 r^{k-1}r^{l-1}\left(2\log\frac{1}{r}\right)^{1-2s}rdr,\\
&= 2\pi\sum_{k=1}^{\infty}\frac{k^2}{k^{2s}}|\hat{f}(k)|^2\int_0^1 r^{2k-1}\left(2\log\frac{1}{r}\right)^{1-2s}dr,\\
&= \pi\sum_{k=1}^{\infty}\frac{k^2}{k^{2s}}|\hat{f}(k)|^2 k^{2s-2}\Gamma(2-2s),\\
&= \pi\Gamma(2-2s)\sum_{n=1}^{\infty}|\hat{f}(k)|^2.
\end{aligned}$$

Therefore and hence,

$$|f_s(0)|^2+\frac{1}{\pi\Gamma(2-2s)}\int_{\mathbb{D}}|f_s'(z)|^2\left(2\log\frac{1}{|z|}\right)^{1-2s}dA(z) = \|f_s\|^2_{PL^2(\mathbb{D};s)}.$$

Thus, we established the Littlewood-Paley Identity for $PL^2(\mathbb{D};s)$. □

Let's assume for brevity,

$$C_{PL^2} = \frac{2^{1-2s}}{\pi\Gamma(2-2s)}, \tag{4.2}$$

with this constant, from now, we will consider the following formatting of the $PL^2(\mathbb{D};s)$.

$$\|f_s\|^2_{PL^2(\mathbb{D};s)} = |f_s(0)|^2+C_{PL^2}\int_{\mathbb{D}}|f_s'(z)|^2\left(\log\frac{1}{|z|}\right)^{1-2s}dA(z).$$

Notation 4.2. *Lets have some useful notation for $s\le\frac{1}{2}$, that will show their presence in the rest of the paper frequently. We have following notation*

$$dH(z) = C_{PL^2}\left(\log\frac{1}{|z|}\right)^{1-2s}dA(z), \tag{4.3}$$

where C_{PL^2} is being defined in (4.2).

4.2 Nevanlinna Counting Function

In this section, we develop the theory and the notion of Nevanlinna counting function for $PL^2(\mathbb{D}; s)$. Before we shall define Nevanlinna counting function for $PL^2(\mathbb{D}; s)$, we shall first recall the notion of *partial counting function*, usual *Nevanlinna counting function* and *change of variables.*

4.2.1 Partial Counting Function

Definition 4.3. *Suppose ϕ is a holomorphic map on $\mathbb{D}$. Let $\{z_j(w)\}_{j\geq 1}$ denotes the points of the pre-image of $\phi^{-1}\{w\}$ where $w \in \mathbb{C} \setminus \phi(0)$ and sequenced in the increasing moduli order along with its multiple repetition, if necessary. Denote $\#(r, w)$ as the number of these point in the disc $r\mathbb{D}$ for $0 \leq r < 1$. Then the* partial counting functions *for ϕ is defined as follows:*

$$N_\phi(r, w) = \sum_{j=1}^{\#(r,w)} \log \frac{r}{|z_j|}. \tag{4.4}$$

4.2.2 Nevanlinna Counting Function

We give the definition of *Nevanlinna counting function* that is prominent in the study of H^2 as follows:

Definition 4.4 (Nevanlinna counting function for H^2)**.** *Suppose ϕ is holomorphic on $\mathbb{D}$. Then N_ϕ is the Nevanlinna counting function of ϕ given as follows:*

$$N_\phi(w) = \begin{cases} \sum_{w\in\phi^{-1}(w)} \log \dfrac{1}{|z|} & \text{if } w \in \phi(\mathbb{D}), \\ 0 & \text{if } w \notin \phi(\mathbb{D}). \end{cases} \tag{4.5}$$

With (4.5) as the definition of the (original) Nevanlinna counting function, we can have following relation with partial counting function that is defined in (4.4):

$$
\begin{aligned}
N_\phi(w) &= N_\phi(1, w), \\
&= \sum_j \log \frac{1}{|z_j(w)|}.
\end{aligned} \tag{4.6}
$$

We have an important inequality called as *Littlewood's inequality* and is given as follows:

Proposition 4.5 (Littlewood's inequality for H^2). *Suppose for all $w \in \mathbb{D} \setminus \phi(0)$, then Nevanlinna counting function N_ϕ satisfies following inequality:*

$$
N_\phi(w) \leq \log \left| \frac{1 - \overline{\phi(0)} w}{\phi(0) - w} \right|. \tag{4.7}
$$

Proof. See [45, Section 4.2.]. □

We are now, in good shape to define the *Nevanlinna counting function* for $PL^2(\mathbb{D}; s)$ as follows.

Definition 4.6 (Nevanlinna counting function for $PL^2(\mathbb{D}; s)$). *Consider $w \in \mathbb{C} \setminus \phi(0)$, and $r \in [0, 1)$ and $s \leq \frac{1}{2}$ we write:*

$$
\begin{aligned}
\mathcal{N}_{\phi,s}(w) &= \sum_{j=1}^{\#(r,w)} \log \left(\frac{r}{|z_j(w)|} \right)^{1-2s}, \quad \textit{and} \\
\mathcal{N}_{\phi,s}(w) &= \mathcal{N}_{\phi,s}(1, w) = \sum_{j=1} \left[\log \left(\frac{1}{|z_j(w)|} \right) \right]^{1-2s}.
\end{aligned}
$$

Here, again $\{z_j(w)\}$ represents the multiplicity sequence of ϕ-pre-images of w. On the account of brevity, $\mathcal{N}_s$ implies $\mathcal{N}_{\phi,s}$. The function $\mathcal{N}_s$ is called as the Nevanlinna counting function *for $PL^2(\mathbb{D}; s)$.*

Useful to note that following equalities are true:

$$
\mathcal{N}_0(w) = N_\phi(w), \quad \text{and} \tag{4.8}
$$

$$\mathcal{N}_{\frac{1}{2}}(w) = \#(r, w). \tag{4.9}$$

We have new version of Littlewood's inequality involving the Nevanlinna counting function $\mathcal{N}_s$ for $PL^2(\mathbb{D}; s)$ which is given in PROPOSITION 4.7 as follows:

Proposition 4.7 (Littlewood's inequality for $PL^2(\mathbb{D}; s)$)**.** *For all $s \leq 0$ and $w \in \mathbb{D} \setminus \phi(0)$, the Nevanlinna counting function $\mathcal{N}_s$ enjoys the following inequality:*

$$\mathcal{N}_s(w) \leq \left(\log\left|\frac{1-\overline{\phi(0)}w}{\phi(0)-w}\right|\right)^{1-2s}. \tag{4.10}$$

Proof. The proof for (4.10) is divided in two parts and given as follows:

1. First case, suppose $s = 0$, then

$$\begin{aligned}
\mathcal{N}_s(w) &= \mathcal{N}_0(w) && (\textit{use } s = 0), \\
&= N_\phi(w) && (\textit{use } (4.8)), \\
&\leq \log\left|\frac{1-\overline{\phi(0)}w}{\phi(0)-w}\right| && (\textit{use } (4.7)).
\end{aligned}$$

   That is, this situation that corresponds to the original Littlewood's inequality presented in (4.7) in PROPOSITION 4.5.

2. Second and last case, on the other hand if $s < 0$, then $\gamma > 1$ where $\gamma = 1 - 2s$. Use this along with the proof of Proposition 6.3 of [45], and thus we are done with the proof of (4.10).

□

One can realize that the behaviour of Nevanlinna counting function for $PL^2(\mathbb{D}; s)$ in (4.10) is similar to the original Nevanlinna counting function that we observe in Littlewood's inequality in (4.7).

4.2.3 Change of Variable Formula

We recall the change of variable formula from H^2 as follows:

Proposition 4.8 (Change of variable in H^2). *If g is a positive measurable function on $\mathbb{D}$, then:*

$$\int_{\mathbb{D}}(g\circ\phi)|\phi'|^2 d\lambda_1 = 2\int_{\mathbb{D}} gN_\phi dA,$$

where

$$d\lambda_1(w) = \log\frac{1}{|w|^2}dA(w).$$

Proof. See [45, Section 4.3]. □

In the spirit of Proposition 4.8, we can have the following Lemma 4.9 which shows the change of variable formula for $PL^2(\mathbb{D};s)$. In actual sense, the proof is similar to the original one, therefore we do not include the proof for this here.

Lemma 4.9 (Change of variable in $PL^2(\mathbb{D};s)$). *If g is a positive measurable function on $\mathbb{D}$, then:*

$$\int_{\mathbb{D}}(g\circ\phi)|\phi'|^2 dH = C_{PL^2}\int_{\mathbb{D}} g\mathcal{N}_s dA, \tag{4.11}$$

where the constant C_{PL^2} is already defined in (4.2) *and dH is defined in* (4.3).

The direct consequence of Lemma 4.9 is the following corollary.

Corollary 4.10. *If $f_s \in PL^2(\mathbb{D};s)$, then*

$$\|f_s\circ\phi\|^2_{PL^2(\mathbb{D};s)} = |f_s(\phi(0))|^2 + C_{PL^2}\int_{\mathbb{D}}|f_s'|^2\mathcal{N}_s dA. \tag{4.12}$$

Proposition 4.11. *If s is strictly negative that is $s<0$, then*

$$\frac{1}{2}\mathcal{N}_s(r,w) = s(2s-1)\int_0^r \frac{\mathcal{N}_0(t,w)}{\left(\log\frac{r}{t}\right)^{1+2s}}dt. \tag{4.13}$$

Proof. Use $\gamma = 1 - 2s$ in [45, Proposition 6.6] and the result is established. $\square$

Theorem 4.12 (Sub-harmonic property of $\mathcal{N}_s$)**.** *Suppose $s < 0$, then the sub-harmonic property of $\mathcal{N}_s$ is given as follows for g being holomorphic on a plane region Ω:*

$$\text{Area}(\Delta)\mathcal{N}_s(g(a)) \leq \int_\Delta \mathcal{N}_s(g(w))dA(w), \tag{4.14}$$

where Δ is an open disc in $\Omega \setminus g^{-1}(\phi(0))$ with center a.

Proof. See [45, Corollary 6.7.]. $\square$

4.3 Upper Bound of Essential norm

4.3.1 Upper Bound

Proposition 4.13. *Suppose $\mathcal{T}$ is a bounded linear operator on a Hilbert space H. Let $\{\mathcal{P}_n\}$ be a sequence of compact self-adjoint operator on H, and write $\mathfrak{P}_n = I - \mathcal{P}_n$. Suppose $\|\mathfrak{P}_n\| = 1$ for each n, and $\|\mathfrak{P}_n x\| \to 0$ for all $x \in H$. Then:*

$$\|\mathcal{T}\|_e = \lim_{n\to\infty} \|\mathcal{T}\mathfrak{P}_n\|,$$

where $\|\mathcal{T}\|_e$ represents the essential norm *of the operator $\mathcal{T}$.*

Proof. See [45, Proposition 5.1.]. $\square$

Lemma 4.14. *Suppose $f_s \in PL^2(\mathbb{D}; s)$ and consider it has a zero of order n at the origin, then for each $z \in \mathbb{D}$, following is true:*

1. *For $s \geq 0$, following is true:*

$$|f_s(z)|^2 \leq \frac{|z|^{2n}}{n^{2s}(1 - |z|^2)} \|f_s\|^2_{PL^2(\mathbb{D};s)}. \tag{4.15}$$

2. For $s \geq 0$, following is true:

$$|f_s'(z)|^2 \leq \frac{|z|^{2n}}{n^{2s-2}\left(1-|z|^2\right)} \|f_s\|^2_{PL^2(\mathbb{D};s)}. \tag{4.16}$$

Proof. We give the proof of inequalities presented above as follows. Consider following series representation for f_s,

$$f_s(z) = \sum_{k=n}^{\infty} \frac{\hat{f}(k)}{k^s} z^k.$$

1. Upon considering the above power series expansion, we have following immediately:

$$\begin{aligned}
|f(z)| &= \left| \sum_{k=n}^{\infty} \frac{\hat{f}(k)}{k^s} z^k \right|, \\
&\leq \sqrt{\sum_{k=n}^{\infty} \left| \frac{\hat{f}(k)}{k^s} \right|^2} \sqrt{\sum_{k=n}^{\infty} |z|^{2k}} && (\textit{use Cauchy-Schwarz}), \\
&\leq \sqrt{\frac{1}{n^{2s}} \sum_{k=n}^{\infty} |\hat{f}(k)|^2} \frac{|z|^n}{\sqrt{1-|z|^2}}, \\
&\leq \frac{|z|^n}{n^s \sqrt{1-|z|^2}} \|f_s\|_{PL^2(\mathbb{D};s)}.
\end{aligned}$$

   Squaring the above inequality establishes (4.15).

2. Similarly, we have following,

$$\begin{aligned}
|f'(z)| &= \left| \sum_{k=n}^{\infty} \frac{\hat{f}(k)}{k^{s-1}} z^{k-1} \right|, \\
&\leq \sqrt{\sum_{k=n}^{\infty} \left| \frac{z^{k-1}}{k^{s-1}} \right|^2} \sqrt{\sum_{k=n}^{\infty} |\hat{f}(k)|^2} && (\textit{use Cauchy-Schwarz}), \\
&\leq \sqrt{\sum_{k=1}^{\infty} \frac{|z|^{2(k+n-1)}}{(k+n-1)^{2s-2}}} \|f_s\|_{PL^2(\mathbb{D};s)}, \\
&\leq |z|^{n-1} \sqrt{\frac{1}{n^{2s-2}} \sum_{k=1}^{\infty} |z|^{2k}} \|f_s\|_{PL^2(\mathbb{D};s)},
\end{aligned}$$

$$\leq \frac{|z|^n}{n^{s-1}\sqrt{1-|z|^2}}\|f_s\|_{PL^2(\mathbb{D};s)}.$$

Squaring the above inequality establishes (4.16).

Thus, our proof is done. $\square$

Lemma 4.15. *The Nevanlinna counting function $\mathcal{N}_s$ for $PL^2(\mathbb{D};s)$ satisfies the following inequality:*

$$\int_{\mathbb{D}} \mathcal{N}_s dA \leq \frac{1}{C_{PL^2}}. \tag{4.17}$$

Proof. Considering $\phi \equiv z$ in (4.12), the following is obtained:

$$\|\phi\|^2_{PL^2(\mathbb{D};s)} = |\phi(0)|^2 + C_{PL^2}\int_{\mathbb{D}} \mathcal{N}_s dA.$$

Hence, with easy transportation in above:

$$C_{PL^2}\int_{\mathbb{D}} \mathcal{N}_s dA = \|\phi\|^2_{PL^2(\mathbb{D};s)} - |\phi(0)|^2 \leq 1 - |\phi(0)|^2 \leq 1.$$

Therefore, dividing above inequality by C_{PL^2} yields the desired inequality (4.17). $\square$

Now, we have the main goal of the chapter given in THEOREM 4.16.

Theorem 4.16 (Principal Goal 1)**.** *The upper bound of the essential norm of C_ϕ in $PL^2(\mathbb{D};s)$ represented by $\|C_\phi\|_e$ is given as follows:*

$$\|C_\phi\|_e^2 \leq \limsup_{|w|\to 1^-} \frac{\mathcal{N}_s(w)}{\left(\log\frac{1}{|w|}\right)^{1-2s}}, \tag{4.18}$$

where $\mathcal{N}_s$ is the Nevanlinna counting function for $PL^2(\mathbb{D};s)$ and $0 < s \leq \frac{1}{2}$.

Proof. We will implement the PROPOSITION 4.13 with $\mathcal{P}_n$ operator defined as follows on $f_s(z) = \sum_{k=1}^{\infty} \frac{\hat{f}(k)}{k^s} z^k \in PL^2(\mathbb{D}; s)$:

$$\mathcal{P}_n f_s(z) = \sum_{k=1}^{n} \frac{\hat{f}(k)}{k^s} z^k. \tag{4.19}$$

Notice that $\mathcal{P}_n$ defined in (4.19) is the orthogonal projection of $PL^2(\mathbb{D}; s)$ onto the closed subspace which is spanned by the monomials $z, \cdots, z^n$ (see inner product in (1.20)), it is self adjoint and compact. Since $\mathfrak{P}_n$ defined as follows:

$$\mathfrak{P}_n = I - \mathcal{P}_n,$$

is the complimentary projection, therefore its norm is 1. Hence, in the light of PROPOSITION 4.13, we have

$$\|C_\phi\|_e = \lim_{n\to\infty} \|C_\phi \mathfrak{P}_n\|,$$

where $\|C_\phi\|_e$ is the essential norm of C_ϕ over $PL^2(\mathbb{D}; s)$. Therefore, by COROLLARY 4.10, we have following

$$\|C_\phi \mathfrak{P}_n f_s\|^2_{PL^2(\mathbb{D};s)} = |\mathfrak{P}_n f_s(\phi(0))|^2 + C_{PL^2} \int_{\mathbb{D}} |\mathfrak{P}_n f_s'|^2 \mathcal{N}_s dA.$$

Since $\|f_s\|_{PL^2(\mathbb{D};s)} \leq 1$, the same reasoning applies for $\mathfrak{P}_n f_s$. Since $\mathfrak{P}_n f_s$ has a zero of order n at the origin, therefore by incorporating results (4.15) and (4.16) of LEMMA 4.14, we get following:

$$
\begin{aligned}
\|C_\phi \mathfrak{P}_n\|^2_{PL^2(\mathbb{D};s)} &\leq \left[C_{PL^2} \sup \int_{\mathbb{D}\setminus r\mathbb{D}} |\mathfrak{P}_n f_s'|^2 \mathcal{N}_s dA\right] \\
&\quad + \left[C_{PL^2} \frac{r^{2n}}{n^{2s-2}(1-r^2)} \int_{r\mathbb{D}} \mathcal{N}_s dA\right] + \left[\frac{|\phi(0)|^{2n}}{n^{2s}(1-|\phi(0)|^2)}\right], \\
&\leq \left[C_{PL^2} \sup \int_{\mathbb{D}\setminus r\mathbb{D}} |(\mathfrak{P}_n f_s')|^2 \mathcal{N}_s dA\right] \\
&\quad + \left[C_{PL^2} \frac{r^{2n}}{n^{2s-2}(1-r^2) C_{PL^2}}\right] + \left[\frac{|\phi(0)|^{2n}}{n^{2s}(1-|\phi(0)|^2)}\right], \\
&= \left[C_{PL^2} \sup \int_{\mathbb{D}\setminus r\mathbb{D}} |(\mathfrak{P}_n f_s')|^2 \mathcal{N}_s dA\right] \\
&\quad + \left[\frac{r^{2n}}{n^{2s-2}(1-r^2)}\right] + \left[\frac{|\phi(0)|^{2n}}{n^{2s}(1-|\phi(0)|^2)}\right].
\end{aligned}
$$

Note that, in the second step above we used (4.17) for the middle term in the *RHS*. Letting $n \to \infty$ and denoting $\mathfrak{B}$ as the unit ball for $PL^2(\mathbb{D}; s)$, and denoting

$$
\mathfrak{h}(w) = \frac{\mathcal{N}_s(w)}{\left(\log \dfrac{1}{|w|}\right)^{1-2s}}.
$$

Proceeding further, we get

$$
\|C_\phi\|_e^2 \leq C_{PL^2} \sup_{\mathfrak{B}} \int_{\mathbb{D}\setminus r\mathbb{D}} |f_s'| \mathcal{N}_s dA = \sup_{\mathfrak{B}} \int_{\mathbb{D}} |f_s'|^2 \mathfrak{h} dH \leq \left[\sup_{r\leq|w|<1} \mathfrak{h}(w)\right] \sup_{\mathfrak{B}} \int_{\mathbb{D}} |f_s'|^2 dH \leq \sup_{r\leq|w|<1} \mathfrak{h}(w).
$$

The last statement follows by the Littlewood-Paley Identity for $PL^2(\mathbb{D}; s)$ and by approaching r to 1, consequently establishes the upper bound of the $\|C_\phi\|_e$, that is (4.18). $\square$

4.4 Angular Derivative

4.4.1 Definition

Definition 4.17. *By supposing ϕ to have a finite* angular derivative *at $\zeta \in \partial\mathbb{D}$, we mean that the following difference quotient,*

$$\frac{\phi(z) - \omega}{z - \zeta},$$

is finite as z approaches to ζ non-tangentially for some $\omega \in \partial\mathbb{D}$. If this limit exists, it is called as the angular derivative *of ϕ at ζ and is represented by $\phi'(\zeta)$.*

1. ϕ has angular derivative at ζ,

2. ϕ has a non-tangential limit of modulus 1 at ζ, and the complex derivative ϕ' has a finite non-tangential limit at ζ.

3. $\phi'(\zeta) = \phi^*(\zeta)\overline{\zeta}d$, provided following is true:

$$\liminf \left\{ \frac{1 - |\phi(z)|}{1 - |z|} \text{ when } z \to \zeta \text{ unrestrictedly in } \mathbb{D} \right\} = d < \infty. \tag{4.20}$$

In accordance with the Schwarz lemma, the quantity d present in (4.20) cannot be 0. With that being noted, this imply that the angular derivative can *never* be 0. Follow [45] for more details to this.

4.4.2 Angular Derivative and Upper Bound Estimate

For $\omega \in \partial\mathbb{D}$ and $\kappa \geq 0$ we have:

$$\varepsilon_\kappa(\omega) = \sum_{\zeta \in \mathcal{E}(\phi,\omega)} \left\{ \frac{1}{|\phi'(\zeta)|^\kappa} \right\},$$

where $|\phi'(\zeta)|$ denotes the magnitude of the angular derivative of ϕ at ζ, if it exists and ∞ if it does not exist. We define $\mathcal{E}(\phi, \omega)$ as follows:

$$\mathcal{E}(\phi, \omega) = \{\zeta \in \partial\mathbb{D}, \phi^*(\zeta) = \omega\}.$$

We recall useful results from H^2 as follows:

Proposition 4.18. *Let $\|C_\phi\|_{\mathrm{e}}$ be the essential norm of C_ϕ on H^2. We have following inequality when $\kappa = 1$:*

$$\sup_{\omega \in \partial\mathbb{D}} \{\varepsilon_1(\omega)\} \leq \|C_\phi\|_{\mathrm{e}}^2.$$

Proof. See [45, Theorem 3.3.]. □

Note that, Schwarz lemma assures that $|\phi'(z)|$ is bounded away from 0 on $\partial\mathbb{D}$ by a constant that is dependent only on $\phi(0)$. Therefore, supposing that $0 < \kappa < \kappa'$, then the Julia-Caratheòdory theorem forces to have,

$$\varepsilon_{\kappa'}(\omega) \leq \varphi_0 \cdot \varepsilon_\kappa(\omega),$$

where this 'φ_0' depends only on κ, κ' and $\phi(0)$. Therefore, with the advantage of Proposition 4.18, the quantity $\varepsilon_1(\omega)$ is bounded on the unit circle. Hence, the same is valid for $\varepsilon_\kappa(\omega)$ for all $\kappa \geq 1$.

We recall famous results from H^2 in Proposition 4.19, that will be used later in the paper.

Proposition 4.19. *Recall $\|C_\phi\|_{\mathrm{e}}$ as the essential norm of C_ϕ on H^2, then:*

$$\|C_\phi\|_{\mathrm{e}}^2 = \limsup_{|w| \to 1^-} \frac{N_\phi(w)}{\log \dfrac{1}{|w|}}. \tag{4.21}$$

In fact, in particular, C_ϕ is compact on H^2 if and only if $\lim_{|w|\to 1^-} \dfrac{N_\phi(w)}{\log \dfrac{1}{|w|}} = 0$.

Proof. See [45, Section 5]. $\square$

Proposition 4.20. *Suppose ϕ is an inner function, then following is true:*

$$\|C_\phi\|_{\mathrm{e}} = \sqrt{\left[\frac{1+|\phi(0)|}{1-|\phi(0)|}\right]}. \tag{4.22}$$

Proof. See [45, Theorem 2.5]. $\square$

Now, we prove our second and last principal goal of this chapter.

Theorem 4.21 (Principal Goal 2)**.** *The essential norm of C_ϕ on $PL^2(\mathbb{D}; s)$ satisfies the following in relation with the angular derivative of ϕ for $0 < s \le \frac{1}{2}$:*

$$\|C_\phi\|_e^2 \le \|C_\phi\|_{\mathrm{e}}^2 \sup_{\zeta\in\partial\mathbb{D}} \left\{|\phi'(\zeta)|^{-2s}\right\}. \tag{4.23}$$

Proof. We begin the proof of having the upper bound for $\|C_\phi\|_e$ as follows: For each $w \in \mathbb{D} \setminus (0, \phi(0))$, we will pick $z = z(w) \in \phi^{-1}\{w\}$, which is of minimum modulus, therefore the following is true:

$$\frac{\mathcal{N}_s(w)}{\left(\log \dfrac{1}{|w|}\right)^{1-2s}} \le \frac{\mathcal{N}_0(w)}{\log \dfrac{1}{|w|}} \left[\frac{\log \dfrac{1}{|z|}}{\log \dfrac{1}{|w|}}\right]^{-2s}. \tag{4.24}$$

Now we will choose $\omega \in \partial\mathbb{D}$ and a sequence of points $\{\omega_n\} \in \mathbb{D}$ such that $\omega_n \to \omega$ and

$$\frac{\mathcal{N}_s(w_n)}{\left(\log \dfrac{1}{|\omega_n|}\right)^{1-2s}} \to \limsup_{|w|\to 1^-} \frac{\mathcal{N}_s(w)}{\left(\log \dfrac{1}{|\omega_n|}\right)^{1-2s}}.$$

By passing to a further sub-sequence (supposing that is required), we may also assume that the sequence of selected pre-images $\{z(w_n)\}$ also converges. Under the influence of Schwarz lemma, its limit must be a point ζ which lives in $\partial\mathbb{D}$. In accordance with the last part of the Julia-Carathèodory theorem:

$$\limsup_{z\to\zeta}\left[\frac{\log\dfrac{1}{|z|}}{\log\dfrac{1}{\phi(z)}}\right]=|\phi'(\zeta)|^{-1}.$$

Hence by (4.24),

$$\begin{aligned}\limsup_{|w|\to 1^-}\frac{\mathcal{N}_s(w)}{\left(\log\dfrac{1}{|w|}\right)^{1-2s}} &\le \left[\limsup_{|w|\to 1^-}\frac{\mathcal{N}_0(w)}{\left(\log\dfrac{1}{|w|}\right)}\right]\left[|\phi'(\zeta)|^{-1}\right]^{2s},\\ &= \left[\limsup_{|w|\to 1^-}\frac{N_\phi(w)}{\left(\log\dfrac{1}{|w|}\right)}\right]\left[|\phi'(\zeta)|^{-1}\right]^{2s} && (\textit{use } (4.8)),\\ &= \|C_\phi\|_{\mathrm{e}}^2\left[|\phi'(\zeta)|^{-1}\right]^{2s}, && (\textit{use } (4.21)).\end{aligned}$$

With this, we have achieved the upper bound on the essential norm of C_ϕ over $PL^2(\mathbb{D};s)$. □

Corollary 4.22. *If ϕ is an inner function, then following inequality is true for $0<s\le\frac{1}{2}$:*

$$\|C_\phi\|_e^2\le\left[\frac{1+|\phi(0)|}{1-|\phi(0)|}\right]\left[\sup_{\zeta\in\partial\mathbb{D}}\left\{|\phi'(\zeta)|^{-2s}\right\}\right]. \tag{4.25}$$

Proof. Use (4.22) in (4.23) and hence, (4.25) is established. □

CHAPTER 5

MITTAG-LEFFLER REPRODUCING KERNEL HILBERT SPACE

One can immediately realize that functions e^z, $\cos z$ and $\cosh z$ can be recovered from E_q defined in (1.4) upon putting $q = 1$ against z and $q = 2$ against $-z^2$ and z^2 respectively. Follow [16] for more on this. In the present avenue of the discussion, lets replace z by $\alpha^q z$ in (1.4) where both α and q are positive and we get following eventually:

$$E_q(\alpha^q z) = \sum_{k=0}^{\infty} \frac{(\alpha^q z)^k}{\Gamma(qk+1)},$$

and in the spirit of defining the kernel represented by $K_q^{[\alpha]}(\cdot,\cdot)$, we have following:

$$K_q^{[\alpha]}(z,w) = E_q\left(\alpha^q z\overline{w}\right) = \sum_{n\geq 0} \frac{(\alpha^q z\overline{w})^n}{\Gamma(qn+1)}. \tag{5.1}$$

The present chapter serves the purpose of studying the Mittag-Leffler reproducing kernel Hilbert space represented here by $M_\alpha^\star$ with its associated kernel defined in (5.1). Note that, we loose no generality if choose any particular value of α, in fact, in particular if $\alpha = 1$ in (5.1), we get the standard Mittag-Leffler kernel function $K_q^{[1]}(z,w)$ which was introduced by Rosenfeld et.al. in [34] (see Definition 4).

5.1 Basics

Consider following measure over the complex plane $\mathbb{C}$ for any positive parameter α,

$$dJ_\alpha(z)^{[2]} = dJ_\alpha(z) := \frac{\alpha}{\pi q}|z|^{\frac{2}{q}-2}e^{-\alpha|z|^{\frac{2}{q}}}dA(z), \tag{5.2}$$

where dA is the Euclidean area measure on the complex plane $\mathbb{C}$. We see that a calculation with polar coordinates shows that $d\mathbf{J}_\alpha(z)$ defined and introduced in (5.2) is a probability measure.

5.1.1 Orthonormal Basis for $M_\alpha^\star(2)$

Theorem 5.1. *For $f \in M_\alpha^\star(2)$, the norm f is as follows:*

$$\|f\|^2_{M_\alpha^\star(2)} := \int_{\mathbb{C}} |f(z)|^2 d\mathbf{J}_\alpha(z)^{[2]}. \tag{5.3}$$

Proof. The proof can be provided by standard and considerable adjustments to the proof of [34, Theorem 3.2.]. □

Lemma 5.2. *The collection of orthonormal basis for $M_\alpha^\star(2)$ is $\left\{\sqrt{\frac{\alpha^{nq}}{\Gamma(qn+1)}}z^n\right\}_n$.*

Proof. Follow [34]. □

Now we present the equivalent norm representation for $f \in M_\alpha^\star(2)$.

Theorem 5.3 (Equivalent norm representation of $M_\alpha^\star(2)$)**.** *Let $f(z) = \sum_{n=0}^\infty \hat{f}(n)z^n \in M_\alpha^\star(2)$, then:*

$$\|f\|^2_{M_\alpha^\star} = |f(0)|^2 + \frac{q}{\pi}\int_{\mathbb{C}} |f'(z)|^2 \left(\int_{|z|^{\frac{q}{2}}}^\infty \frac{e^{-\alpha t}}{t}dt\right) dA(z). \tag{5.4}$$

Proof. The *LHS* of (5.4) is of course, $\|f\|^2_{M^\star_\alpha}=\sum_{n=0}^{\infty}|\hat{f}(n)|^2\Gamma(qn+1)/\alpha^{nq}$. Note that $f(0)=\hat{f}(0)$, therefore $|f(0)|^2=|\hat{f}(0)|^2$. Now we will focus on the remaining part of *RHS* in (5.4) as follows:

$$
\begin{aligned}
\int_{\mathbb{C}}|f'(z)|^2\left(\int_{|z|^{\frac{q}{2}}}^{\infty}\frac{e^{-\alpha t}}{t}dt\right)dA(z) &= \int_{\mathbb{C}}\left|\sum_{n=1}^{\infty}n\hat{f}(n)z^{n-1}\right|^2\left(\int_{|z|^{\frac{q}{2}}}^{\infty}\frac{e^{-\alpha t}}{t}dt\right)dA(z),\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nm\hat{f}(n)\overline{\hat{f}(m)}2\pi\delta_{nm}\int_0^{\infty}r^{n-1}r^{m-1}\left(\int_{r^{\frac{q}{2}}}^{\infty}\frac{e^{-\alpha t}}{t}dt\right)rdr,\\
&= 2\pi\sum_{n=1}^{\infty}n^2|\hat{f}(n)|^2\int_0^{\infty}\int_{r^{\frac{q}{2}}}^{\infty}r^{2n-1}\frac{e^{-\alpha t}}{t}dtdr,\\
&= 2\pi\sum_{n=1}^{\infty}n^2|\hat{f}(n)|^2\int_0^{\infty}\frac{e^{-\alpha t}}{t}\left(\int_0^{t^{\frac{q}{2}}}r^{2n-1}dr\right)dt \quad (\text{by } \textit{Fubini}),\\
&= 2\pi\sum_{n=1}^{\infty}n^2|\hat{f}(n)|^2\int_0^{\infty}\frac{e^{-\alpha t}}{t}\frac{t^{n\alpha}}{2n}dt,\\
&= \pi\sum_{n=1}^{\infty}n|\hat{f}(n)|^2\int_0^{\infty}t^{nq-1}e^{-\alpha t}dt,\\
&= \pi\sum_{n=1}^{\infty}n|\hat{f}(n)|^2\frac{\Gamma(nq)}{\alpha^{nq}},\\
&= \frac{\pi}{q}\sum_{n=1}^{\infty}|\hat{f}(n)|^2\left[nq\Gamma(nq)\frac{1}{\alpha^{nq}}\right].
\end{aligned}
$$

Hence after dividing above by $\frac{\pi}{q}$ in the last step above, we get following:

$$
\frac{q}{\pi}\int_{\mathbb{C}}|f'(z)|^2\left(\int_{|z|^{\frac{q}{2}}}^{\infty}\frac{e^{-\alpha t}}{t}dt\right)dA(z)=\sum_{n=1}^{\infty}|\hat{f}(n)|^2\frac{\Gamma(qn+1)}{\alpha^{nq}}. \tag{5.5}
$$

Here, we use the property of *Gamma function*, given as $\Gamma(s+1)=s\Gamma(s)$. Thus, setting $s=nq$, establishes $nq\Gamma(nq)=\Gamma(nq+1)$. Now, adding $|f(0)|^2$ to (5.5) yields following:

$$
\begin{aligned}
|f(0)|^2+\frac{q}{\pi}\int_{\mathbb{C}}|f'(z)|^2\left(\int_{|z|^{\frac{q}{2}}}^{\infty}\frac{e^{-\alpha t}}{t}dt\right)dA(z) &= |f(0)|^2+\sum_{n=1}^{\infty}|\hat{f}(n)|^2\frac{\Gamma(nq+1)}{\alpha^{nq}},\\
&= |\hat{f}(0)|^2+\sum_{n=1}^{\infty}|\hat{f}(n)|^2\frac{\Gamma(nq+1)}{\alpha^{nq}},\\
&= \sum_{n=0}^{\infty}|\hat{f}(n)|^2\frac{\Gamma(nq+1)}{\alpha^{nq}},\\
&= \|f\|^2_{M^\star_\alpha}.
\end{aligned} \tag{5.6}
$$

We used $|f(0)|^2 = |\hat{f}(0)|^2$ in (5.6). Therefore, (5.4) is established. $\square$

The $M_\alpha^\star(2)$ consists of all entire functions f such that $f \in L^2(\mathbb{C}, d\mathbf{J}_\alpha(z))$. Now, we will show that $M_\alpha^\star(2)$ is a closed subspace of $L^2(\mathbb{C}, d\mathbf{J}_\alpha(z))$.

Theorem 5.4. *$M_\alpha^\star(2)$ is a closed subspace of $L^2(\mathbb{C}, d\mathbf{J}_\alpha(z))$.*

Proof. Recall that a subspace M of a Hilbert space $\mathcal{H}$ is a closed subspace if and only if it is itself a Hilbert space (with the same inner product). Applying this fact to our case, provides the desired result. $\square$

5.1.2 Reproducing Kernel

Definition 5.5. *Consider both $\alpha > 0$ and $q > 0$, the reproducing kernel for $M_\alpha^\star(2)$ represented by $K_q^{[\alpha]}(z,w)$ is given as follows:*

$$K_q^{[\alpha]}(z,w) = E_q\left(\alpha^q z\overline{w}\right) = \sum_{n\geq 0} \frac{(\alpha^q z\overline{w})^n}{\Gamma(qn+1)}.$$

With the abuse of notation, we often prefer to go for $K_{q,w}^{[\alpha]}$ for a fixed $w \in \mathbb{C}$ to imply $K_q^{[\alpha]}(z,w)$. For any fixed $w \in \mathbb{C}$, the mapping $f \mapsto f(w)$ is a bounded linear functional on $M_\alpha^\star(2)$, due to the mean value theorem. By the Riesz representation theorem in THEOREM 1.2:

$$\langle f, K_{q,w}^{[\alpha]} \rangle = f(w) \qquad \forall f \in M_\alpha^\star(2).$$

Lemma 5.6. *If $f \in M_\alpha^\star(2)$, then $\dfrac{f}{K_{q,w}^{[\alpha]}} \in M_\alpha^\star(2)$.*

Proof. We begin by considering the inner product for $M_\alpha^\star(2)$ as follows:

$$\begin{aligned}\left|\left\langle \frac{f}{K_{q,w}^{[\alpha]}}, K_{q,w}^{[\alpha]} \right\rangle\right| &= \left|\int_{\mathbb{C}} \frac{f(z)}{K_{q,w}^{[\alpha]}(z)} K_{q,w}^{[\alpha]}(z) d\mathbf{J}_\alpha(z)\right|,\\ &= \left|\int_{\mathbb{C}} f(z) d\mathbf{J}_\alpha(z)\right|,\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{C}} |f(z)| d\mathbf{J}_\alpha(z), \\
&\leq \sqrt{\int_{\mathbb{C}} |f(z)|^2 d\mathbf{J}_\alpha(z)} \sqrt{\int_{\mathbb{C}} d\mathbf{J}_\alpha(z)}, && \textit{(use Cauchy-Schwarz)} \\
&= \|f\| < \infty && (\textit{as } f \in M_\alpha^\star(2)).
\end{aligned}
$$

□

We recall following inequality from [18, Remark 2.3] which will be extensively implemented in further discussion here.

Lemma 5.7. *With both q and $\alpha > 0$, there exist positive constants C_1 and C_2 such that following is true:*

$$C_1 e^{t^{\frac{1}{q}}} \leq |E_q(t)| \leq C_2 e^{t^{\frac{1}{q}}}.$$

Corollary 5.8. *Let $f \geq 0$ and $f \in L^1\left(\mathbb{C}, d\mathbf{J}_\alpha(z)^{[\mathbf{2}]}\right)$, then for any $z \in \mathbb{C}$, following is true:*

$$\int_{\mathbb{C}} f(z-w) d\mathbf{J}_\alpha(w)^{[\mathbf{2}]} = \int_{\mathbb{C}} f(w) \frac{1}{E_q(\alpha^q |z-w|^2)} d\mathbf{J}_\alpha(w)^{[\mathbf{2}]}, \tag{5.7}$$

and also,

$$\int_{\mathbb{C}} f(z+w) d\mathbf{J}_\alpha(w)^{[\mathbf{2}]} = \int_{\mathbb{C}} f(w) \frac{K_q^{[\boldsymbol{\alpha}]}(2z, w)}{E_q(\alpha^q |w-z|^2)} d\mathbf{J}_\alpha(w)^{[\mathbf{2}]}. \tag{5.8}$$

Proof. We will provide the proof of (5.7) as follows:

$$
\begin{aligned}
\int_{\mathbb{C}} f(z-w) d\mathbf{J}_\alpha(w)^{[\mathbf{2}]} &= \frac{\alpha}{\pi q} \int_{\mathbb{C}} f(z-w) |w|^{\frac{2}{q}-2} e^{-\alpha |w|^{\frac{2}{q}}} dA(w), \\
&= \frac{\alpha}{\pi q} \int_{\mathbb{C}} \frac{f(z-w)}{K_q^{[\boldsymbol{\alpha}]}(z,w)} K_q^{[\boldsymbol{\alpha}]}(z,w) |w|^{\frac{2}{q}-2} e^{-\alpha |w|^{\frac{2}{q}}} dA(w), \\
&= \frac{f(0)}{K_q^{[\boldsymbol{\alpha}]}(z,z)} \qquad (\textit{apply } \textsc{Lemma } 5.6, \textit{ then reproducing property}), \\
&= \int_{\mathbb{C}} \frac{f(w)}{K_q^{[\boldsymbol{\alpha}]}(z-w, z-w)} d\mathbf{J}_\alpha(w)^{[\mathbf{2}]},
\end{aligned}
$$

$$= \int_{\mathbb{C}} f(w) \frac{1}{E_q(\alpha^q |z-w|^2)} d\mathbf{J}_\alpha(w)^{[\mathbf{2}]}.$$

Thus, with this we have established the relation presented in (5.7). Now, we will show the relation given in (5.8) and the whole procedure will be same as what we have done above.

$$\begin{aligned}
\int_{\mathbb{C}} f(z+w) d\mathbf{J}_\alpha(w)^{[\mathbf{2}]} &= \frac{\alpha}{\pi q} \int_{\mathbb{C}} f(z+w) |w|^{\frac{2}{q}-2} e^{-\alpha |w|^{\frac{2}{q}}} dA(w), \\
&= \frac{\alpha}{\pi q} \int_{\mathbb{C}} \frac{f(z+w)}{K_q^{[\alpha]}(z,w)} K_q^{[\alpha]}(z,w) |w|^{\frac{2}{q}-2} e^{-\alpha |w|^{\frac{2}{q}}} dA(w), \\
&= \frac{f(2z)}{K_q^{[\alpha]}(z,z)} \qquad (\textit{apply } \text{Lemma 5.6}, \textit{ then reproducing property}), \\
&= \int_{\mathbb{C}} \frac{f(w)}{K_q^{[\alpha]}(z-w, z-w)} K_q^{[\alpha]}(2z, w) d\mathbf{J}_\alpha(w)^{[\mathbf{2}]}, \\
&= \int_{\mathbb{C}} f(w) \frac{K_q^{[\alpha]}(2z,w)}{E_q(\alpha^q |w-z|^2)} d\mathbf{J}_\alpha(w)^{[\mathbf{2}]}.
\end{aligned}$$

□

Lemma 5.9. *Define following quantity which one can see a relation between l and $N \in \mathbb{Z}_+ \cup \{0\}$,*

$$\lambda_q = \left[\Gamma(qN+1) \sum_{l=0}^{N} \frac{1}{\Gamma(ql+1)\Gamma(qN-ql+1)} \right]^{\frac{1}{qN}}. \tag{5.9}$$

Suppose $\alpha > 0$ and $\beta \in \mathbb{R}$, then following is true:

$$\int_{\mathbb{C}} \left| K_q^{[\beta]}(z,a) \right| d\mathbf{J}_\alpha(z) = K_q^{[\alpha]} \left(\frac{\beta a}{\lambda_q \alpha}, \frac{\beta a}{\lambda_q \alpha} \right). \tag{5.10}$$

Proof. In order to proof (5.10), first we realize first that, in the light of (5.9), following relation holds:

$$K_q^{[\beta]}(z,a) = \left(K_q^{[\alpha]}(a,z) \right)^2.$$

It follows from the definition of the reproducing kernel that

$$K_q^{[\alpha]}(a,a)=\int_{\mathbb{C}}\left|K_q^{[\alpha]}(a,z)\right|^2 d\mathbf{J}_\alpha(z).$$

Now, replacing a by $\dfrac{\beta a}{\lambda_q \alpha}$, yields the desired result. $\square$

Now, consider $\alpha>0$ and $p>0$, we use the notation L^p_α to denote the space of Lebesgue measurable functions f on $\mathbb{C}$ such that the function $f(z)|z|^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}\in L^p(\mathbb{C},dA)$. With the following measure,

$$d\mathbf{J}_\alpha(z)^{[\mathbf{p}]}:=\frac{p\alpha}{2\pi q}|z|^{\frac{p}{q}-p}e^{-\frac{p\alpha}{2}|z|^{\frac{2}{q}}}dA(z). \tag{5.11}$$

we write $f\in L^p_\alpha$ as following:

$$\|f\|^p_{M^\star_\alpha(p)}:=\int_{\mathbb{C}}|f(z)|^p d\mathbf{J}_\alpha(z)^{[\mathbf{p}]}.$$

Again, with $\alpha>0$ and $p=\infty$, we use the notation L^∞_α to denote the space of Lebesgue measurable functions f on $\mathbb{C}$ such that

$$\|f\|_{M^\star_\alpha(\infty)}=\operatorname{esssup}\left\{|f(z)|e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}\text{ for } z\in\mathbb{C}\right\}<\infty.$$

Its important to note that f is entire under the consideration. Following is the isometry relation in $M^\star_\alpha(p)$ that will be used in complex interpolation of $M^\star_\alpha(p)$ further.

Theorem 5.10 (Isometry in $M^\star_\alpha(p)$)**.** *Let $\zeta\in\mathbb{C}\setminus\{0\}$, then define ζ_q as following:*

$$\zeta_q=\zeta^{\frac{1}{q}-1},\quad \textit{and}\quad f_\zeta(z):=\zeta_q^{1-\frac{2}{p}}f(\zeta z).$$

Then, $f(z)\to f_\zeta(z)$ is an isometry from $M^\star_\alpha(p)$ onto $M^\star_{|\zeta|^{\frac{2}{q}}\alpha}(p)$ for $0<p\le\infty$.

Proof. We have following calculation:

$$\begin{aligned}
\|f_\zeta(z)\|^p_{M^\star_{|\zeta|^{\frac{2}{q}}\alpha}} &= \frac{p\alpha|\zeta|^{\frac{2}{q}}}{2\pi q}\int_{\mathbb{C}}|f_\zeta(z)|^p|z|^{\frac{p}{q}-p}e^{-\frac{p|\zeta|^{\frac{2}{q}}\alpha}{2}|z|^{\frac{2}{q}}}dA(z),\\
&= \frac{p\alpha|\zeta|^{\frac{2}{q}}}{2\pi q}\int_{\mathbb{C}}\left|\zeta_q^{1-\frac{2}{p}}f\left(\zeta z\right)\right|^p|z|^{\frac{p}{q}-p}e^{-\frac{p\alpha}{2}|\zeta z|^{\frac{2}{q}}}dA(z),\\
&= \frac{p\alpha|\zeta|^{\frac{2}{q}}}{2\pi q}\int_{\mathbb{C}}\left|\frac{\zeta^{\frac{1}{q}-1}}{\zeta^{(\frac{1}{q}-1)\frac{2}{p}}}f(\zeta z)\right|^p|z|^{\frac{p}{q}-p}e^{-\frac{p\alpha}{2}|\zeta z|^{\frac{2}{q}}}dA(z),\\
&= \frac{p\alpha|\zeta|^{\frac{2}{q}}}{2\pi q}\int_{\mathbb{C}}\left|\frac{\zeta^{\frac{1}{q}-1}}{\zeta^{(\frac{1}{q}-1)\frac{2}{p}}}\right|^p|f(\zeta z)|^p|z|^{\frac{p}{q}-p}e^{-\frac{p\alpha}{2}|\zeta z|^{\frac{2}{q}}}dA(z),\\
&= \frac{p\alpha|\zeta|^{\frac{2}{q}}}{2\pi q}\int_{\mathbb{C}}\frac{|\zeta|^{\frac{p}{q}-p}}{|\zeta|^{\frac{2}{q}-2}}|f(\zeta z)|^p|z|^{\frac{p}{q}-p}e^{-\frac{p\alpha}{2}|\zeta z|^{\frac{2}{q}}}dA(z),\\
&= \frac{p\alpha}{2\pi q}\int_{\mathbb{C}}|f(w)|^p|w|^{\frac{p}{q}-p}e^{-\frac{p\alpha}{2}|w|^{\frac{2}{q}}}dA(w), \qquad\qquad (\textit{where } w=\zeta z)\\
&= \|f\|^p_{M^\star_\alpha}.
\end{aligned}$$

□

Theorem 5.11. *For $f \in M^\star_\alpha$ and $0 < p < \infty$, the point evaluation inequality is satisfied as:* $|f(z)| \leq \|f\|_{M^\star_\alpha(p)}\sqrt{K_q^{[\alpha]}(z,z)}$.

Proof. In the consideration of $0 < p < \infty$, following is achieved:

$$|f(0)|^p \leq \frac{p\alpha}{2\pi q}\int_{\mathbb{C}}|f(z)|^p|z|^{\frac{p}{q}-p}e^{-\frac{\alpha p}{2}|z|^{\frac{2}{q}}}dA(z),$$

due to the sub-harmonicity of $|f|^p$ and integration in polar coordinates. Now, consider function F in the following format,

$$F(w) := f(z-w)\frac{K_q^{[\alpha]}(w,z)}{\sqrt{K_q^{[\alpha]}(z,z)}}, \quad \text{where } f \in M^\star_\alpha(p) \text{ and any } z \in \mathbb{C}.$$

In the light of the fact that $|F(0)|^p \leq \|F\|^p_{M^\star_\alpha(p)}$, we proceed as follows:

$$\begin{aligned}
\left| f(z)\frac{K_q^{[\alpha]}(0,z)}{\sqrt{K_q^{[\alpha]}(z,z)}}\right|^p &\leq \frac{p\alpha}{2\pi q}\int_{\mathbb{C}}\left| f(z-w)\frac{K_q^{[\alpha]}(w,z)}{\sqrt{K_q^{[\alpha]}(z,z)}}\right|^p |w|^{\frac{p}{q}-p}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}dA(w),\\
|f(z)|^p &\leq \frac{p\alpha}{2\pi q}\int_{\mathbb{C}}\left| f(z-w)K_q^{[\alpha]}(w,z)\right|^p |w|^{\frac{p}{q}-p}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}dA(w),\\
&= \frac{p\alpha}{2\pi q}\int_{\mathbb{C}}|f(z-w)|^p\left|K_q^{[\alpha]}(w,z)\right|^p |w|^{\frac{p}{q}-p}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}dA(w),\\
&= \frac{p\alpha}{2\pi q}\int_{\mathbb{C}}|f(z-w)|^p\left[K_q^{[\alpha]}(w,z)\overline{K_q^{[\alpha]}(w,z)}\right]^{\frac{p}{2}} |w|^{\frac{p}{q}-p}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}dA(w),\\
&= \frac{p\alpha}{2\pi q}\int_{\mathbb{C}}|f(z-w)|^p\left[K_q^{[\alpha]}(w,z)K_q^{[\alpha]}(z,w)\right]^{\frac{p}{2}} |w|^{\frac{p}{q}-p}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}dA(w),\\
&= \frac{p\alpha}{2\pi q}\int_{\mathbb{C}}G(z,w)\left[K_q^{[\alpha]}(z,w)\right]^{\frac{p}{2}} |w|^{\frac{p}{q}-p}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}dA(w).
\end{aligned}$$

We have the function $G(z,w) = |f(z-w)|^p\left[K_q^{[\alpha]}(w,z)\right]^{\frac{p}{2}}$ and now we use the reproducing property of $M^\star_\alpha(\frac{p}{2})$ to have: $|f|^p \leq |f(0)|^p\left[\sqrt{K_q^{[\alpha]}(z,z)}\right]^p \leq \|f\|^p_{M^\star_\alpha(p)}\left[\sqrt{K_q^{[\alpha]}(z,z)}\right]^p$. Thus, in conclusion, after taking $1/p$ power above, we have: $|f| \leq \|f\|_{M^\star_\alpha(p)}\sqrt{K_q^{[\alpha]}(z,z)}$, as desired. □

Proposition 5.12. *Suppose $0 < p < \infty$ and put $f_r(z) := f(rz)$, then following holds:*

1. $\|f_r - f\|_{M^\star_\alpha(p)} \to 0$ *as* $r \to 1^-$.

2. *There is a sequence of polynomials* $\{p_n\}$ *of polynomials such that* $\lim_{n\to\infty}\|p_n - f\|_{M^\star_\alpha(p)} = 0$.

Proof. 1. We recall the basic result from the usual L^p-spaces theory as follows:

$$\lim_{n\to\infty}\int_X |g_n - g|^p d\mu = 0 \iff \lim_{n\to\infty}\int_X |g_n|^p d\mu = \int_X |g|^p d\mu,$$

where $\{g_n\}_n$ and g lives in $L^p(X, d\mu)$. Now, letting $f \in M^\star_\alpha$ and then we proceed as follows:

$$\|f_r\|^p_{M^\star_\alpha(p)} = \int_{\mathbb{C}} |f_r(z)|^p\, d\mathbf{J}_\alpha(z)^{[p]},$$

$$= \frac{p\alpha}{2\pi q}\int_{\mathbb{C}}|f(rz)|^p|z|^{\frac{p}{q}-p}e^{-\frac{p\alpha}{2}|z|^{\frac{2}{q}}}dA(z),$$
$$= \frac{p\alpha}{2\pi q}\int_{\mathbb{C}}|f(w)|^p\left|\frac{w}{r}\right|^{\frac{p}{q}-p}e^{-\frac{p\alpha}{2}|\frac{w}{r}|^{\frac{2}{q}}}\frac{dA(w)}{r^2},$$
$$= \frac{p\alpha}{2\pi q r^{2+\frac{p}{q}-p}}\int_{\mathbb{C}}|f(w)|^p|w|^{\frac{p}{q}-p}e^{-\frac{p\alpha}{2}|\frac{w}{r}|^{\frac{2}{q}}}dA(w),$$
$$= \frac{p\alpha}{2\pi q r^{2+\frac{p}{q}-p}}\int_{\mathbb{C}}|f(w)|^p|w|^{\frac{p}{q}-p}e^{-\frac{p\alpha}{2}|w|^{\frac{2}{q}}}e^{-\frac{p\alpha}{2}|w|^{\frac{2}{q}}\left[\frac{1}{r}^{\frac{2}{q}}-1\right]}dA(w).$$

Since, we have $0 < r < 1$, we have following: $e^{-\frac{p\alpha}{2}|w|^{\frac{2}{q}}\left[\frac{1}{r}^{\frac{2}{q}}-1\right]} \leq 1,\ \forall w \in \mathbb{C}$. An application of Lebesgue Dominated Convergence Theorem shows that following is true:

$$\|f_r\|_{M^\star_\alpha(p)} \to \|f\|_{M^\star_\alpha(p)} \qquad \text{as } n \to \infty.$$

Combining this argument with the previous arguments yields the desired result.

2. In the light of the THEOREM 5.11 we have $M^\star_\beta(2) \subset M^\star_\alpha(p)$, for β from $(r^{\frac{2}{q}}\alpha, \alpha)$ where obviously $0 < r < 1$. Due to this set containment relationship, we obtain following:

$$\|g\|_{M^\star_\alpha(p)} \leq C\|g\|_{M^\star_\beta(2)},$$

where we pick $g \in M^\star_\beta(2)$. We consider p_n as the n^{th} Taylor-polynomial for f_r and therefore

$$\|f_r - p_n\|_{M^\star_\alpha(p)} \leq C\|f_r - p_n\|_{M^\star_\beta(2)} \to 0 \qquad \text{as } n \to \infty.$$

□

Theorem 5.13. *Let $q > 0$ and $\frac{1}{q} < 1$. Suppose $0 < p \leq p' < \infty$, then $\|f\|_{M^\star_\alpha(p')} \leq C\|f\|_{M^\star_\alpha(p)}$, where $C = \left(\frac{p'}{p}\right)^{\frac{1}{p}}$.*

Proof. Since, $\frac{1}{q} < 1$, this means that $|z|^{\frac{p'}{q}-p'} \leq |z|^{\frac{p}{q}-p}$. For any entire function $f \in M_\alpha^\star(p)$, we consider the integral in the following manner:

$$\begin{aligned}
\|f\|_{M_\alpha^\star(p')}^{p'} &= \frac{p'\alpha}{2\pi q}\int_{\mathbb{C}} |f|^{p'}|z|^{\frac{p'}{q}-p'} e^{-\frac{\alpha p'}{2}|z|^{\frac{2}{q}}}\, dA(z),\\
&= \frac{p'\alpha}{2\pi q}\int_{\mathbb{C}} |f|^{p'-p}|f|^{p}|z|^{\frac{p}{q}-p} e^{-\frac{\alpha p'}{2}|z|^{\frac{2}{q}}}\, dA(z),\\
&\leq \frac{p'\alpha}{2\pi q}\int_{\mathbb{C}} \|f\|_{M_\alpha^\star(p)}^{p'-p}\left[\sqrt{K_q^{[\alpha]}(z,z)}\right]^{p'-p} |f|^{p}|z|^{\frac{p}{q}-p} e^{-\frac{\alpha p'}{2}|z|^{\frac{2}{q}}}\, dA(z),\\
&= \frac{p'\alpha}{2\pi q}\|f\|_{M_\alpha^\star(p)}^{p'-p}\int_{\mathbb{C}} \left[\sqrt{E_q(\alpha^q|z|^2)}\right]^{p'-p} |f|^{p}|z|^{\frac{p}{q}-p} e^{-\frac{\alpha p'}{2}|z|^{\frac{2}{q}}}\, dA(z),\\
&\leq \frac{p'\alpha}{2\pi q}\|f\|_{M_\alpha^\star(p)}^{p'-p}\int_{\mathbb{C}} \left[\sqrt{e^{(\alpha^q|z|^2)^{\frac{1}{q}}}}\right]^{p'-p} |f|^{p}|z|^{\frac{p}{q}-p} e^{-\frac{\alpha p'}{2}|z|^{\frac{2}{q}}}\, dA(z),\\
&= \frac{p'\alpha}{2\pi q}\|f\|_{M_\alpha^\star(p)}^{p'-p}\int_{\mathbb{C}} \left[e^{\left(\alpha^q|z|^2\right)^{\frac{1}{q}}}\right]^{\frac{p'-p}{2}} |f|^{p}|z|^{\frac{p}{q}-p} e^{-\frac{\alpha p'}{2}|z|^{\frac{2}{q}}}\, dA(z),\\
&= \frac{p'\alpha}{2\pi q}\|f\|_{M_\alpha^\star(p)}^{p'-p}\int_{\mathbb{C}} \left[e^{\alpha|z|^{\frac{2}{q}}}\right]^{\frac{p'-p}{2}} |f|^{p}|z|^{\frac{p}{q}-p} e^{-\frac{\alpha p'}{2}|z|^{\frac{2}{q}}}\, dA(z),\\
&= \frac{p'\alpha}{2\pi q}\|f\|_{M_\alpha^\star(p)}^{p'-p}\int_{\mathbb{C}} e^{\alpha|z|^{\frac{2}{q}}\left(\frac{p'-p}{2}\right)} |f|^{p}|z|^{\frac{p}{q}-p} e^{-\frac{\alpha p'}{2}|z|^{\frac{2}{q}}}\, dA(z),\\
&= \frac{p'}{p}\frac{p\alpha}{2\pi q}\|f\|_{M_\alpha^\star(p)}^{p'-p}\int_{\mathbb{C}} |f|^{p}|z|^{\frac{p}{q}-p} e^{-\frac{\alpha p}{2}|z|^{\frac{2}{q}}}\, dA(z),\\
&= \frac{p'}{p}\|f\|_{M_\alpha^\star(p)}^{p'}.
\end{aligned}$$

Now, taking the $1/p'$ power of the above inequality yields the desired result. Now we will see that the inclusion as discussed above is proper. Let $I : M_\alpha^\star(p) \to M_\alpha^\star(p')$ be the identity map under the assumption that $M_\alpha^\star(p') = M_\alpha^\star(p)$. Therefore, I is one-to-one, and onto. By the open mapping theorem, there must exist a constant $C > 0$ such that

(5.12) $$\frac{1}{C}\|f\|_{M_\alpha^\star(p)} \leq \|z\|_{M_\alpha^\star(p')} \leq C\|f\|_{M_\alpha^\star(p)}.$$

On the other hand, with the help of Stirling's approximation of Gamma function, we have following:

$$
\begin{aligned}
\|z^n\|^p_{M^\star_\alpha(p)} &= p\alpha\left(\frac{2}{\alpha p}\right)^{\frac{npq}{2}+\frac{p}{2}-\frac{pq}{2}+q}\Gamma\left(\frac{npq}{2}+\frac{p}{2}-\frac{pq}{2}+q\right),\\
&\approx p\alpha\left(\frac{2}{\alpha p}\right)^{\frac{npq}{2}+\frac{p}{2}-\frac{pq}{2}+q}\sqrt{2\pi\left[\left(\frac{2}{\alpha p}\right)^{\frac{npq}{2}+\frac{p}{2}-\frac{pq}{2}+q}\right]}\left(\frac{\left(\frac{2}{\alpha p}\right)^{\frac{npq}{2}+\frac{p}{2}-\frac{pq}{2}+q}}{e}\right)^{\left(\frac{2}{\alpha p}\right)^{\frac{npq}{2}+\frac{p}{2}-\frac{pq}{2}+q}}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\|z^n\|^{p'}_{M^\star_\alpha(p')} &= p'\alpha\left(\frac{2}{\alpha p'}\right)^{\frac{np'q}{2}+\frac{p'}{2}-\frac{p'q}{2}+q}\Gamma\left(\frac{np'q}{2}+\frac{p'}{2}-\frac{p'q}{2}+q\right),\\
&\approx p'\alpha\left(\frac{2}{\alpha p'}\right)^{\frac{np'q}{2}+\frac{p'}{2}-\frac{p'q}{2}+q}\sqrt{2\pi\left[\left(\frac{2}{\alpha p'}\right)^{\frac{np'q}{2}+\frac{p'}{2}-\frac{p'q}{2}+q}\right]}\left(\frac{\left(\frac{2}{\alpha p'}\right)^{\frac{np'q}{2}+\frac{p'}{2}-\frac{p'q}{2}+q}}{e}\right)^{\left(\frac{2}{\alpha p'}\right)^{\frac{np'q}{2}+\frac{p'}{2}-\frac{p'q}{2}+q}}.
\end{aligned}
$$

It is then obvious that there is no positive constant C with the property that (5.12) is satisfied. □

Definition 5.14. *We define $f^\infty_{M^\star_\alpha}$ as the space of entire functions such that following is satisfied by f:*

$$\lim_{z\to\infty}\frac{f(z)}{\sqrt{K^{[\alpha]}_q(z,z)}}=0.$$

Theorem 5.15. *The space $f^\infty_{M^\star_\alpha}$ is separable in contrast to $M^\star_\alpha(\infty)$.*

Proof. Note that $f^\infty_{M^\star_\alpha}$ is a closed subspace of $M^\star_\alpha(\infty)$. Also, for the set of polynomials, $f^\infty_{M^\star_\alpha}$ is the closure in $M^\star_\alpha(\infty)$. This imply that $f^\infty_{M^\star_\alpha}$ is separable but the same cannot happen in $M^\star_\alpha(\infty)$. □

Lemma 5.16. *For any positive parameters α and γ, the set of functions of the form*

$$f(z)=\sum_{k=1}^{n}c_kK^{[\alpha]}_q(z,w_k),$$

is dense in $M^{\star}_{\alpha}(p)$ *for* $0<p<\infty$ *and also in* $f^{\infty}_{M^{\star}_{\alpha}}$*.*

Proof. Since, the points w_k are arbitrary, we may assume that $\gamma=\alpha$. The result is obvious when $p=2$. In fact, if a function $h\in M^{\star}_{\alpha}(2)$ is orthogonal to each function $f(z)=K^{[\alpha]}_{q}(z,w)$, then $h(w)=0$ for every w. In general, with the help of THEOREM 5.11, we can find a positive parameter β such that $M^{\star}_{\beta}(2)\subset M^{\star}_{\alpha}(p)$ continuously, say

$$\|f\|_{M^{\star}_{\alpha}(p)}\leq C\|f\|_{M^{\star}_{\beta}(2)},$$

for all $f\in M^{\star}_{\beta}(2)$, (in fact every β from $(0,\alpha)$ works). Now, if f is a polynomial and $\{w_q,\dots,w_n\}\in\mathbb{C}$, then

$$\begin{aligned}\left\|f(z)-\sum_{k=1}^{n}c_kK^{[\alpha]}_{q}(z,w_k)\right\|_{M^{\star}_{\alpha}(p)} &\leq C\left\|f(z)-\sum_{k=1}^{n}c_kK^{[\alpha]}_{q}(z,w_k)\right\|_{M^{\star}_{\beta}(2)},\\ &= C\left\|f(z)-\sum_{k=1}^{n}c_kK^{[\alpha]}_{q}\left(z,\left(\frac{\alpha}{\beta}\right)^{q}w_k\right)\right\|_{M^{\star}_{\beta}(2)}.\end{aligned}$$

Combining this with the density of the functions $\sum_{k=1}^{n}c_kK^{[\alpha]}_{q}(z,u_k)\in M^{\star}_{\beta}(2)$, we conclude that every polynomial can be approximated in the norm topology of $M^{\star}_{\alpha}(p)$ by functions of the form $\sum_{k=1}^{n}c_kK^{[\alpha]}_{q}(z,u_k)$. Since the polynomials are dense in $M^{\star}_{\alpha}(p)$, we have proved the result for $M^{\star}_{\alpha}(p)$ for $0<p<\infty$. The proof for $f^{\infty}_{M^{\star}_{\alpha}}$ is similar. □

We have interesting technical lemma for $0<q<2$ that are inspired by the study of [6].

Lemma 5.17. *For* $0<q<2$*, and* $0<p<\infty$*, then there exists a positive constant* C *such that following is true for all entire functions* f*:*

1.

$$\int_{\mathbb{C}}|f(w)|^{p}|w|^{\frac{2}{q}-2}e^{-\beta|w|^{\frac{2}{q}}}dA(w)\geq C\int_{\mathbb{C}}|f(w)|^{p}e^{-\beta|w|^{\frac{2}{q}}}dA(w);$$

2.

$$\int_{\mathbb{C}} |f(w)|^p |w|^{\frac{2}{q}-2} e^{-\beta|w|^{\frac{2}{q}}} dA(w) \leq C \int_{|w|\geq 1} |f(w)|^p |w|^{\frac{2}{q}-2} e^{-\beta|w|^{\frac{2}{q}}} dA(w)$$

Proof. Considering $0 < q < 2$.

1. First part result follows from the subharmonicity of $|f(z)|^p$ and the maximum principle, there is a positive constant C such that,

$$\int_{|w|\leq 1} |f(w)|^p dA(w) \leq C \int_{|w|\leq 1} |f(w)|^p |w|^{\frac{2}{q}-2} dA(w),$$

   for all entire functions f and $0 < p < \infty$. Since, $e^{-\beta|w|^{\frac{2}{q}}}$ is both bounded above and bounded below on $|w| \leq 1$, we deduce that

$$\int_{\mathbb{C}} |f(w)|^p |w|^{\frac{2}{q}-2} e^{-\beta|w|^{\frac{2}{q}}} dA(w) \geq C' \int_{\mathbb{C}} |f(w)|^p e^{-\beta|w|^{\frac{2}{q}}} dA(w),$$

   which is our desired result.

2. Second part result follows from the subharmonicity of $|f(z)|^p |z|^{\frac{2}{q}-2}$ and integration in polar coordinates.

□

5.2 Integral operators

We recall previously known results from theory of integral operators.

Lemma 5.18. *Say, $1 \leq p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. If an integral operator given as $Tf(x) = \int_X H(x,y) f(y) d\mu(y)$, is bounded on $L^p(X, d\mu)$, then its adjoint with mapping given as $T^* : L^{p'}(X, d\mu) \to L^{p'}(X, d\mu)$, is the integral operator given as $T^* f(x) = \int_X \overline{H(y,x)} f(y) d\mu(y)$.*

Proof. See [19] or [53]. □

Lemma 5.19. *Say $H(x,y)$ is a positive kernel and $Tf(x) = \int_X H(x,y)f(y)d\mu(y)$, is the associated integral operator. Suppose $p \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. If there exist a positive function $h(x)$ and positive constants C_1 and C_2 such that*

$$\int_X H(x,y)h(y)^{p'}d\mu(y) \leq C_1 h(x)^{p'},\ \forall x \in X,$$
$$\int_X H(x,y)h(x)^{p}d\mu(x) \leq C_2 h(y)^{p},\ \forall y \in X,$$

then the operator T is bounded on $L^p(X,d\mu)$ and $\|T\| \leq C_1^{\frac{1}{p'}} C_2^{\frac{1}{p}}$.

Proof. See [54] or [53]. □

Definition 5.20. *We fix two positive parameters α and β for the rest of this section and we define following integral operators P_α and Q_α over $L^p(\mathbb{C}, d\mathbf{J}_\beta(z))$ as follows:*

$$P_\alpha f(z) = \frac{\alpha}{\beta}\int_{\mathbb{C}} f(w)K_q^{[\alpha]}(z,w)e^{(\beta-\alpha)|w|^{\frac{2}{q}}}d\mathbf{J}_\beta(w), \tag{5.13}$$

and

$$Q_\alpha f(z) = \frac{\alpha}{\beta}\int_{\mathbb{C}} f(w)\left|K_q^{[\alpha]}(z,w)e^{(\beta-\alpha)|w|^{\frac{2}{q}}}\right|d\mathbf{J}_\beta(w). \tag{5.14}$$

Following can easily be retrieved by Lemma *5.18.*

$$P_\alpha^* f(z) = \frac{\alpha}{\beta}e^{(\beta-\alpha)|z|^{\frac{2}{q}}}\int_{\mathbb{C}} f(w)K_q^{[\alpha]}(z,w)d\mathbf{J}_\beta(w), \tag{5.15}$$

and

$$Q_\alpha^* f(z) = \frac{\alpha}{\beta}e^{(\beta-\alpha)|z|^{\frac{2}{q}}}\int_{\mathbb{C}} f(w)\left|K_q^{[\alpha]}(z,w)\right|d\mathbf{J}_\beta(w). \tag{5.16}$$

Lemma 5.21. *If P_α is bounded on $L^p\left(\mathbb{C}, d\mathbf{J}_\beta(z)^{[\mathbf{2}]}\right)$, then $\alpha p \leq 2\beta$.*

Proof. We will use $f_{x,k}(z) = z^k e^{-x|z|^{\frac{2}{q}}}$. See [8, 52, 53] for this. Its natural to deduce the following calculations:

$$\int_{\mathbb{C}} |f_{x,k}(z)|^p d\mathbf{J}_\beta(z)^{[\mathbf{2}]} = \frac{\beta}{(\beta+px)^{\frac{pqk}{2}+1}}\Gamma\left(\frac{pqk}{2}+1\right).$$

With the following definition of P_α from (5.13), we have:

$$P_\alpha f(z) = \frac{\alpha}{\beta}\int_{\mathbb{C}} f(w) K_q^{[\alpha]}(z,w) e^{(\beta-\alpha)|w|^{\frac{2}{q}}} d\mathbf{J}_\beta(w),$$

and applying above to the function $f_{x,k}$ as follows:

$$\begin{aligned}
P_\alpha f_{x,k}(z) &= \frac{\alpha}{q\pi}\int_{\mathbb{C}} f_{x,k}(w) K_q^{[\alpha]}(z,w)|w|^{\frac{2}{q}-2} e^{-\alpha|w|^{\frac{2}{q}}} dA(w),\\
&= \frac{\alpha}{\alpha+x}\int_{\mathbb{C}} \left[K_q^{[\alpha]}(z,w)w^k\right] d\mathbf{J}_{\alpha+x}(w),\\
&= \frac{\alpha}{\alpha+x}\int_{\mathbb{C}} \left[\sum_{n\geq 0}(\alpha+x)^{qn}\left(\frac{\alpha^q z\overline{w}}{(\alpha+x)^q}\right)^n \frac{1}{\Gamma(qn+1)}\right] w^k d\mathbf{J}_{\alpha+x}(w),\\
&= \frac{\alpha}{\alpha+x}\int_{\mathbb{C}} K_q^{[\alpha+x]}\left(\left(\frac{\alpha}{\alpha+x}\right)^q z, w\right) w^k d\mathbf{J}_{\alpha+x}(w) \qquad (5.17)\\
&= \frac{\alpha^{1+qk}}{(\alpha+x)^{1+qk}} z^k. \qquad (5.18)
\end{aligned}$$

We applied reproducing property of $M^2_{\alpha+x}$ from (5.17) to get to (5.18). Now, we have following

$$\int_{\mathbb{C}} |P_\alpha f_{x,k}|^p d\mathbf{J}_\beta(z) = \left(\frac{\alpha}{\alpha+x}\right)^{pkq+p}\int_{\mathbb{C}} |z|^{pk} d\mathbf{J}_\beta(z) = \left(\frac{\alpha}{\alpha+x}\right)^{pkq+p}\frac{\Gamma\left(\frac{pkq}{2}+1\right)}{\beta^{\frac{pkq}{2}}}.$$

In the consideration of the statement of the lemma, we have a constant $C > 0$ such that following is true:

$$\begin{aligned}
\left(\frac{\alpha}{\alpha+x}\right)^{pkq+p}\frac{\Gamma\left(\frac{pkq}{2}+1\right)}{\beta^{\frac{pkq}{2}}} &\leq C\frac{\beta}{(\beta+px)^{\frac{pqk}{2}+1}}\Gamma\left(\frac{pqk}{2}+1\right),\\
\left(\frac{\alpha}{\alpha+x}\right)^{pkq+p}\frac{1}{\beta^{\frac{pkq}{2}}} &\leq C\frac{\beta}{(\beta+px)^{\frac{pqk}{2}+1}},
\end{aligned}$$

$$\left(\frac{\alpha}{\alpha+x}\right)^{pkq+p} \leq C\frac{\beta\beta^{\frac{pkq}{2}}}{(\beta+px)^{\frac{pqk}{2}+1}}.$$

Fix $x>0$ and raise the above inequality to $\dfrac{2}{pk}$ and letting letting $k\to\infty$ in (5.19) to (5.20), in to have following:

$$\left(\frac{\alpha}{\alpha+x}\right)^{2(q+1/k)} \leq \frac{\beta^{q+2/pk}}{(\beta+px)^{q+2/pk}}, \tag{5.19}$$

$$\left(\frac{\alpha}{\alpha+x}\right)^{2q} \leq \frac{\beta^{q}}{(\beta+px)^{q}}, \tag{5.20}$$

$$\alpha^2(\beta+px) \leq \beta(\alpha+x)^2.$$

Letting $x\to 0$ yields $\alpha p \leq 2\beta$. Similarly, if we let $k=0$ and let $x\to\infty$ in the previous paragraph, the result is $p\geq 1$. This completes the proof of the lemma. □

Lemma 5.22. *If $1<p<\infty$ and P_α is bounded on $L^p\left(\mathbb{C}, d\mathbf{J}_\beta(z)\right)$, then $p\alpha>\beta$.*

Proof. Here, we have $1<p<\infty$ and if P_α is bounded on $L^p(\mathbb{C}, d\mathbf{J}_\beta(z))$, then P_α^* is bounded on $L^{p'}\left(\mathbb{C}, d\mathbf{J}_\beta(z)\right)$ where $\dfrac{1}{p}+\dfrac{1}{p'}=1$. Applying the formula for P_α^* to 1, we have $e^{(\beta-\alpha)|z|^{\frac{2}{q}}} \in L^{p'}\left(\mathbb{C}, d\mathbf{J}_\beta(z)\right)$, thus with the calculation in polar coordinates we have following:

$$\int_0^\infty r^{\frac{2}{q}-1}e^{-(\beta-p'(\beta-\alpha))r^{\frac{2}{q}}}\,dr<\varphi_0<\infty,$$

which implies that $\beta>p'(\beta-\alpha) \implies p\alpha>\beta$. Hence proved. □

Lemma 5.23. *If P_α is bounded on $L^1(\mathbb{C}, d\mathbf{J}_\beta(z))$, then $\alpha \leq \lambda_q\beta\left(\frac{\alpha-\beta}{\beta}\right)^{\frac{q}{2}}$.*

Proof. Consider the following format of the function for $a\in\mathbb{C}$

$$f_a(z)=\frac{K_q^{[\alpha]}(z,a)}{\left|K_q^{[\alpha]}(z,a)\right|}.$$

Then we have following:

$$
\begin{aligned}
P_{\alpha}^{*} f_a(a) &= \frac{\alpha}{\beta} e^{(\beta-\alpha)|a|^{\frac{2}{q}}} \int_{\mathbb{C}} f_a(w) K_q^{[\alpha]}(a,w) d\mathbf{J}_{\beta}(w),\\
&= \frac{\alpha}{\beta} e^{(\beta-\alpha)|a|^{\frac{2}{q}}} \int_{\mathbb{C}} \frac{K_q^{[\alpha]}(w,a)}{\left|K_q^{[\alpha]}(w,a)\right|} K_q^{[\alpha]}(a,w) d\mathbf{J}_{\beta}(w),\\
&= \frac{\alpha}{\beta} e^{(\beta-\alpha)|a|^{\frac{2}{q}}} \int_{\mathbb{C}} \left|K_q^{[\alpha]}(w,a)\right| d\mathbf{J}_{\beta}(w),\\
&= \frac{\alpha}{\beta} e^{(\beta-\alpha)|a|^{\frac{2}{q}}} K_q^{[\beta]} \left(\frac{\alpha a}{\lambda_q \beta}, \frac{\alpha a}{\lambda_q \beta}\right),\\
&\geq C \frac{\alpha}{\beta} e^{(\beta-\alpha)|a|^{\frac{2}{q}}} e^{\beta\left(\frac{\alpha^2|a|^2}{\lambda_q^2\beta^2}\right)^{\frac{1}{q}}}.
\end{aligned}
$$

Use Lemma 5.9 in the above second last step. Since, P_{α}^{*} is bounded on $L^{\infty}(\mathbb{C})$, we have following

$$
\frac{\alpha}{\beta} e^{(\beta-\alpha)|a|^{\frac{2}{q}}} e^{\beta\left(\frac{\alpha^2|a|^2}{\lambda_q^2\beta^2}\right)^{\frac{1}{q}}} \leq \|P_{\alpha}^{*} f\|_{\infty} \leq C\|f\|_{\infty} = C.
$$

Thus we have following after essential calculation from above:

$$
(\beta-\alpha) + \beta\left(\frac{\alpha}{\lambda_q \beta}\right)^{\frac{2}{q}} \leq 0 \implies \alpha \leq \lambda_q \beta \left(\frac{\alpha-\beta}{\beta}\right)^{\frac{q}{2}}.
$$

which is our desired result. □

Lemma 5.24. *Suppose $1 < p \leq 2$ and P_{α} is bounded on $L^p(\mathbb{C}, d\mathbf{J}_{\beta}(z))$, then $p\alpha = 2\beta$.*

Proof. Once again, we consider functions of the form $f_{x,k}(z) = z^k e^{-x|z|^{\frac{2}{q}}}$. We then have following calculations:

$$
\int_{\mathbb{C}} |f_{x,k}(z)|^{p'} d\mathbf{J}_{\beta}(z)^{[\mathbf{2}]} = \frac{\beta}{(\beta + p'x)^{\frac{p'qk}{2}+1}} \Gamma\left(\frac{p'qk}{2}+1\right).
$$

On the other hand, we have following:

$$P_\alpha^* f_{x,k}(z) = \frac{\alpha}{\pi q} e^{(\beta-\alpha)|z|^{\frac{2}{q}}} \int_{\mathbb{C}} K_q^{[\alpha]}(z,w) e^{-x|w|^{\frac{2}{q}}} w^k |w|^{\frac{2}{q}-2} e^{-\alpha|w|^{\frac{2}{q}}} dA(w) = \frac{\alpha^{1+qk}}{(\beta+x)^{1+qk}} z^k e^{(\beta-\alpha)|z|^{\frac{2}{q}}}.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{C}} |P_\alpha^* f_{x,k}(z)|^{p'} \, d\mathbf{J}_\beta(z) &= \left(\frac{\alpha}{\beta+x}\right)^{(1+qk)p'} \frac{\beta}{\pi q} \int_{\mathbb{C}} e^{p'(\beta-\alpha)|z|^{\frac{2}{q}}} |z|^{p'k} |z|^{\frac{2}{q}-2} e^{-\beta|z|^{\frac{2}{q}}} dA(z), \\
&= \left(\frac{\alpha}{\beta+x}\right)^{(1+qk)p'} \frac{\beta}{(\beta - p'(\beta-\alpha))^{\frac{p'qk}{2}+1}} \Gamma\left(\frac{p'qk}{2}+1\right).
\end{aligned}$$

In the consideration of the hypothesis, we have following:

$$\int_{\mathbb{C}} |P_\alpha^* f_{x,k}(z)|^{p'} \, d\mathbf{J}_\beta(z) \leq C \int_{\mathbb{C}} |f_{x,k}(z)|^{p'} d\mathbf{J}_\beta(z),$$

which results into following:

$$\begin{aligned}
\left(\frac{\alpha}{\beta+x}\right)^{(1+qk)p'} \frac{\beta}{(\beta - p'(\beta-\alpha))^{\frac{p'qk}{2}+1}} \Gamma\left(\frac{p'qk}{2}+1\right) &\leq C \frac{\beta}{(\beta+p'x)^{\frac{p'qk}{2}+1}} \Gamma\left(\frac{p'qk}{2}+1\right), \\
\left(\frac{\alpha}{\beta+x}\right)^{2q} &\leq \left(\frac{\beta - p'(\beta-\alpha)}{\beta+p'x}\right)^{q}, \\
&= \left(\frac{p\alpha-\beta}{(p-1)\beta+px}\right)^{q}.
\end{aligned}$$

Reducing above by taking $\frac{1}{q}$ of above, we have following:

$$(p\alpha-\beta)x^2 + \left(2\beta(p\alpha-\beta) - \alpha^p\right)x + \beta^2(p\alpha-\beta) - \alpha^2(p-1)\beta \geq 0.$$

Let $q(x)$ denote the quadratic function on the left-hand side of the above inequality. By LEMMA 5.22, we already have $p\alpha > \beta$, $q(x)$ attains its minimum value at $x_0 = \frac{p\alpha^2 - 2\beta(p\alpha-\beta)}{2(p\alpha-\beta)}$, and since $p \leq 2$, the top part of x_0 is greater than or equal to $p\alpha^2 - 2\alpha\beta p + p\beta^2 = p(\alpha-\beta)^2 \geq 0$.

Also, $q(x) \geq q(x_0)$ for all x and from this, we deduce that the discriminant of $q(x)$ cannot be positive. Therefore, $(p\alpha - 2\beta)^2 \leq 0 \implies p\alpha = 2\beta$. This gives the desired result.

$\square$

Lemma 5.25. *Suppose $2 < p < \infty$ and P_α is bounded on $L^p(\mathbb{C}, d\mathbf{J}_\beta(w))$, then $p\alpha = 2\beta$.*

Proof. Again, given that P_α is bounded on $L^p(\mathbb{C}, d\mathbf{J}_\beta(z))$ then P_α^* is bounded on $L^{p'}(\mathbb{C}, d\mathbf{J}_\beta(z))$ with $1 < p' < 2$ and $\dfrac{1}{p'} + \dfrac{1}{p} = 1$. We have a positive constant C such that

$$\int_{\mathbb{C}} \left| e^{(\beta-\alpha)|z|^{\frac{2}{q}}} \int_{\mathbb{C}} K_q^{[\alpha]}(z,w) \left[f(w) e^{(\alpha-\beta)|z|^{\frac{2}{q}}} \right] d\mathbf{J}_\alpha(w) \right|^{p'} d\mathbf{J}_\beta(z) \leq C \int_{\mathbb{C}} |f(w)|^{p'} d\mathbf{J}_\beta(w),$$

where $f \in L^{p'}(\mathbb{C}, d\mathbf{J}_\beta(z))$. Let $f(z) = g(z) e^{(\beta-\alpha)|z|^{\frac{2}{q}}}$, where $g \in L^{p'}(\mathbb{C}, d\mathbf{J}_{\beta - p'(\beta-\alpha)})$. By, Lemma 5.22, we have $\beta - p'(\beta - \alpha) > 0$. We obtain another positive constant C (independent of g) such that

$$\int_{\mathbb{C}} |P_\alpha g|^{p'} d\mathbf{J}_{\beta - p'(\beta-\alpha)} \leq C \int_{\mathbb{C}} |g|^{p'} d\mathbf{J}_{\beta - p'(\beta-\alpha)},$$

for all $g \in L^{p'}(\mathbb{C}, d\mathbf{J}_{\beta - p'(\beta-\alpha)})$. Since $1 < p' < 2$, it follows from Lemma 5.24, we have $p'\alpha = 2\left(\beta - p'(\beta - \alpha)\right)$. The last equation above is $p\alpha = 2\beta$ and this is our desired result. $\square$

Theorem 5.26. *Recall λ_q and suppose $p = 1$, then following are equivalent:*

1. *Q_α is bounded on $L^p(\mathbb{C}, d\mathbf{J}_\beta(z))$,*
2. *P_α is bounded on $L^p(\mathbb{C}, d\mathbf{J}_\beta(z))$,*
3. *finally, $\alpha \leq \lambda_q \beta \left(\frac{\alpha-\beta}{\beta}\right)^{\frac{q}{2}}$.*

Proof. *(1.)* $\implies$ *(2.)* obvious. *(2.)* $\implies$ *(3.)* by Lemma 5.23. Lastly, *(3.)* $\implies$ *(1.)* from Fubini's theorem and Lemma 5.9. $\square$

Theorem 5.27. *Suppose $1 < p < \infty$, then following are equivalent:*

1. *Q_α is bounded on $L^p(\mathbb{C}, d\mathbf{J}_\beta(z))$,*

2. P_α is bounded on $L^p(\mathbb{C}, d\mathbf{J}_\beta(z))$,

3. finally, $p\alpha = 2\beta$.

Proof. By LEMMA 5.24 and LEMMA 5.25 on $1 < p < \infty$ we have *(2.)* $\implies$ *(3.)* and *(1.)* $\implies$ *(2.)* is still obvious. We shall consider $1 < p < \infty$ and show *(3.)* $\implies$ *(1.)* and we do this with the help of Schur's test. Let $\frac{1}{p'} + \frac{1}{p} = 1$ and consider the positive function

$$h(z) = e^{\delta |z|^{\frac{2}{q}}}.$$

Recall that

$$Q_\alpha f(z) = \int_{\mathbb{C}} H(z,w) f(w) d\mathbf{J}_\beta(w),$$

where

$$H(z,w) = \frac{\alpha}{\beta} \left| K_q^{[\alpha]}(z,w) e^{(\beta-\alpha)|w|^{\frac{2}{q}}} \right|$$

is a positive kernel. We first consider the integrals

$$I(z) = \int_{\mathbb{C}} H(z,w) h(w)^{p'} d\mathbf{J}_\beta(w) \tag{5.21}$$

and assuming $\alpha - p'\delta > 0$ we have following

$$\begin{aligned} I(z) &= \left(\frac{\alpha}{\alpha - p'\delta}\right) K_q^{[\alpha - p'\delta]} \left(\frac{\alpha z}{\lambda_q(\alpha - p'\delta)}, \frac{\alpha \bar{z}}{\lambda_q(\alpha - p'\delta)}\right), \\ &= \left(\frac{\alpha}{\alpha - p'\delta}\right) E_q \left((\alpha - p'\delta)^q \left(\frac{\alpha|z|}{\lambda_q(\alpha - p'\delta)}\right)^2\right), \\ &\le \left(C_1 \frac{\alpha}{\alpha - p'\delta}\right) e^{\alpha - p'\delta \left[\frac{\alpha}{\lambda_q(\alpha - p'\delta)}\right]^{\frac{2}{q}} |z|^{\frac{2}{q}}}. \end{aligned}$$

The above C_1 comes from the following inequality:

$$E_q \left((\alpha - p'\delta)^q \left(\frac{\alpha|z|}{\lambda_q(\alpha - p'\delta)}\right)^2\right) \le C_1 e^{\alpha - p'\delta \left[\frac{\alpha}{\lambda_q(\alpha - p'\delta)}\right]^{\frac{2}{q}} |z|^{\frac{2}{q}}}.$$

Now, if we choose δ so that

$$(\alpha - p'\delta)\left(\frac{\alpha}{\lambda_q\left(\alpha - p'\delta\right)}\right)^{\frac{2}{q}} = p'\delta, \tag{5.22}$$

then one can have

$$\int_{\mathbb{C}} H(z,w)h(w)^{p'}d\mathbf{J}_\beta(w) \leq \left(C\frac{\alpha}{\alpha - p'\delta}\right)h(z)^{p'}. \tag{5.23}$$

In similar ways, we have following for all $w \in \mathbb{C}$,

$$J(w) = \int_{\mathbb{C}} H(z,w)h(z)^p d\mathbf{J}_\beta(z),$$

and assuming that $\beta - p\delta > 0$, then we have following

$$\begin{aligned} J(w) &= \frac{\alpha}{\beta - p\delta}e^{(\beta-\alpha)|w|^{\frac{2}{q}}}K_q^{[\beta - p\delta]}\left(\frac{\alpha w}{\lambda_q(\beta - p\delta)}, \frac{\alpha\overline{w}}{\lambda_q(\beta - p\delta)}\right), \\ &= \frac{\alpha}{\beta - p\delta}e^{(\beta-\alpha)|w|^{\frac{2}{q}}}E_q\left((\beta - p\delta)^q\left(\frac{\alpha|w|}{\beta - p\delta}\right)^2\right), \\ &\leq C_2\frac{\alpha}{\beta - p\delta}e^{(\beta-\alpha)|w|^{\frac{2}{q}}}e^{\beta - p\delta\left[\frac{\alpha}{\lambda_q(\beta - p\delta)}\right]^{\frac{2}{q}}|w|^{\frac{2}{q}}}. \end{aligned}$$

Again, the C_2 above comes in the same way as C_1 came in the above paragraph. If we choose δ so that

$$(\beta - \alpha) + \beta - p\delta\left[\frac{\alpha}{\lambda_q(\beta - p\delta)}\right]^{\frac{2}{q}} = p\delta, \tag{5.24}$$

then we see that

$$\int_{\mathbb{C}} H(z,w)h(z)^p d\mathbf{J}_\beta(z) \leq C\frac{\alpha}{\beta - p\delta}h(w)^p. \tag{5.25}$$

In the view of Schur's test and the estimates in the conditions (5.23) and (5.25) we conclude that the operator Q_α would be bounded on $L^p(\mathbb{C}, d\mathbf{J}_\beta(z))$ provided that we could choose a real δ to satisfy conditions $\alpha - p'\delta > 0$, $\beta - p\delta > 0$ along with (5.22) and (5.24) to be true. These leads to following equality:

$$(\alpha - p'\delta)^{\frac{2}{q}-1} p'\delta = \left(\frac{\alpha}{\lambda_q}\right)^{\frac{2}{q}}, \tag{5.26}$$

and

$$(\beta - p\delta)^{\frac{2}{q}-1} (p\delta - \beta + \alpha) = \left(\frac{\alpha}{\lambda_q}\right)^{\frac{2}{q}}. \tag{5.27}$$

In the view of $p\alpha = 2\beta$ and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, indeed the solution of (5.22), (5.24), (5.26) and (5.27) when solved for δ will be same upon the consistency of the equations and following be satisfied with $\delta = \delta_q$ as the solution:

$$\left(\frac{\alpha}{\lambda_q}\right)^{\frac{2}{q}} = \left(\frac{\alpha - p'\delta_q}{\beta - p\delta_q}\right)^{\frac{2}{q}-1} \left(\frac{p\delta_q - \beta + \alpha}{p'\delta_q}\right)^{-1}. \tag{5.28}$$

This completes the proof of the theorem. □

The δ_q coming from (5.28) will be used in the following theorem and subsequent corollary.

Theorem 5.28. *If $1 \leq p < \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and $p\alpha = 2\beta$, then*

$$\int_{\mathbb{C}} |P_\alpha f|^p d\mathbf{J}_\beta(z) \leq \int_{\mathbb{C}} |Q_\alpha f|^p d\mathbf{J}_\beta(z) \leq C \left[\left(\frac{\alpha}{\alpha - p'\delta_q}\right)^{\frac{1}{p'}} \left(\frac{\alpha}{\beta - p\delta_q}\right)^{\frac{1}{p}} \right] \int_{\mathbb{C}} |f|^p d\mathbf{J}_\beta(z)$$

for all $f \in L^p(\mathbb{C}, d\mathbf{J}_\beta(z))$ and $C = (C_1)^{\frac{1}{p'}} (C_2)^{\frac{1}{p}} > 0$, where C_1 and C_2 comes from the THEOREM *5.27.*

Corollary 5.29. *For any $\alpha > 0$ and $1 \leq p \leq \infty$, P_α is a bounded projection from L^p_α onto $M^\star_\alpha(p)$*

$$\|P_\alpha f\|_{M^\star_\alpha(p)} \leq C\left[\left(\frac{\alpha}{\alpha - p'\delta_q}\right)^{\frac{1}{p'}}\left(\frac{\alpha}{\beta - p\delta_q}\right)^{\frac{1}{p}}\right]\|f\|_{M^\star_\alpha(p)}, \tag{5.29}$$

for all $f \in L^p_\alpha$ and $C = (C_1)^{\frac{1}{p'}}(C_2)^{\frac{1}{p}} > 0$, where C_1 and C_2 comes from the Theorem *5.27.*

Proof. The case $1 \leq p < \infty$ follows from Theorem 5.28 and for $p = \infty$ follows from Lemma 5.9. □

5.3 Duality

In this section, we will identify all the bounded linear functional on the Mittag-Leffler space $M^\star_\alpha$ whenever $0 < p < \infty$ and also for $f^\infty_{M^\star_\alpha}$. Its interesting to know that for $q > 0$, the duality of Mittag-Leffler space depends on the geometric mean of α and β.

Theorem 5.30. *Suppose $\beta > 0$, $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, then dual of $M^\star_\alpha(p)$ is $M^\star_\beta(p')$ under the following integral pairing:*

$$\langle f, g\rangle_\gamma = \frac{\gamma}{\pi q}\lim_{R\to\infty}\int_{|z|<R} f(z)\overline{g(z)}|z|^{\frac{2}{q}-2}e^{-\gamma|z|^{\frac{2}{q}}}\,dA(z), \tag{5.30}$$

where $\gamma = (\alpha\beta)^{\frac{1}{2}}$.

Proof. To begin with, we assume that $g \in M^\star_\beta(p')$ and say F is defined as follows:

$$F(f) = \frac{\gamma}{\pi q}\lim_{R\to\infty}\int_{|z|<R} f(z)\overline{g(z)}|z|^{\frac{2}{q}-2}e^{-\gamma|z|^{\frac{2}{q}}}\,dA(z).$$

We proceed to show that F gives rise to a bounded linear functional on $M^\star_\alpha(p)$ by the assumption that g is is a finite linear combination of kernel functions. If $f(z) = K^{[\gamma]}_q(z, a)$ for some $a \in \mathbb{C}$, then with the help of reproducing property of $K_q^{[\gamma]}(z,a)$, we have following:

$$\begin{aligned} g(a) &= \frac{\gamma}{\pi q}\int_{\mathbb{C}} \overline{K_q^{[\gamma]}(z,a)}g(z)|z|^{\frac{2}{q}-2}e^{-\gamma|z|^{\frac{2}{q}}}dA(z), \\ &= \frac{\gamma}{\pi q}\int_{\mathbb{C}} \overline{f(z)}g(z)|z|^{\frac{2}{q}-2}e^{-\gamma|z|^{\frac{2}{q}}}dA(z). \end{aligned} \tag{5.31}$$

On the other hand, by the reproducing property of $K_q^{[\alpha]}(z,a)$, we have following:

$$\begin{aligned} g(a) &= g\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}}\frac{\gamma^q}{\alpha^q}a\right), \\ &= \frac{\alpha}{\pi q}\int_{\mathbb{C}} K_q^{[\alpha]}\left(\frac{\gamma^q}{\alpha^q}a, z\right) g\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)|z|^{\frac{2}{q}-2}e^{-\alpha|z|^{\frac{2}{q}}}dA(z), \\ &= \frac{\alpha}{\pi q}\int_{\mathbb{C}} \left[\sum_{n\geq 0}\frac{\left(\alpha^q\frac{\gamma^q a}{\alpha^q}\overline{z}\right)^n}{\Gamma(qn+1)}\right] g\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)|z|^{\frac{2}{q}-2}e^{-\alpha|z|^{\frac{2}{q}}}dA(z), \\ &= \frac{\alpha}{\pi q}\int_{\mathbb{C}} \left[\sum_{n\geq 0}\frac{(\gamma^q a\overline{z})^n}{\Gamma(qn+1)}\right] g\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)|z|^{\frac{2}{q}-2}e^{-\alpha|z|^{\frac{2}{q}}}dA(z), \\ &= \frac{\alpha}{\pi q}\int_{\mathbb{C}} K_q^{[\gamma]}(a,z) g\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)|z|^{\frac{2}{q}-2}e^{-\alpha|z|^{\frac{2}{q}}}dA(z), \\ &= \frac{\alpha}{\pi q}\int_{\mathbb{C}} \overline{K_q^{[\gamma]}(z,a)} g\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)|z|^{\frac{2}{q}-2}e^{-\alpha|z|^{\frac{2}{q}}}dA(z), \\ &= \frac{\alpha}{\pi q}\int_{\mathbb{C}} \overline{f(z)} g\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)|z|^{\frac{2}{q}-2}e^{-\alpha|z|^{\frac{2}{q}}}dA(z). \end{aligned} \tag{5.32}$$

Therefore by (5.31) and (5.32), we have following:

$$\int_{\mathbb{C}} f\overline{g}d\mathbf{J}_\gamma = \frac{\alpha}{\pi q}\int_{\mathbb{C}} f(z)\overline{g}\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)|z|^{\frac{2}{q}-2}e^{-\alpha|z|^{\frac{2}{q}}}dA(z). \tag{5.33}$$

This shows that

$$F(f)=\frac{\alpha}{\pi q}\int_{\mathbb{C}} f(z)\overline{g}\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)|z|^{\frac{2}{q}-2}e^{-\alpha|z|^{\frac{2}{q}}}dA(z),$$
$$=\frac{\alpha}{\pi q}\int_{\mathbb{C}}\left[f(z)|z|^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}\right]\left[\overline{g}\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)|z|^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}\right]dA(z),$$

for all functions f of the form $f(z)=\sum_{k=1}^{N} c_k K_q^{[\gamma]}(z,a_k)$, that are dense in $M_\alpha^\star(p)$. Now, it is clear that $g\in L^{p'}\left(\mathbb{C},|z|^{\frac{1}{q}-1}e^{-\frac{\beta}{2}|z|^{\frac{2}{q}}}dA(z)\right)$ is equivalent to the condition that

$$\phi(z)=g\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)\in L^{p'}\left(\mathbb{C},|z|^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}dA(z)\right).$$

Applying the Holder's inequality to have

$$|F(f)|\leq C\|f\|_{M_\alpha^\star(p)}\|\phi\|_{M_\alpha^\star(p')}=C'\|f\|_{M_\alpha^\star(p)}\|g\|_{M_\beta^\star(p')},\tag{5.34}$$

where again, f any linear combination of kernel functions, and C and C' are positive constants. This shows that F defines a bounded linear functional on $M_\alpha^\star(p)$. Next, we assume that $F: M_\alpha^\star(p)\to\mathbb{C}$ is a bounded functional linear in nature. Define g as follows on $\mathbb{C}$

$$\overline{g(w)}=F_z\left(K_q^{[\gamma]}(z,w)\right).$$

It is easy to note that g is entire and now we will show that $g\in M_\beta^\star(p')$ and $F(f)=\langle f,g\rangle_\gamma$ for all f dense in $M_\alpha^\star(p)$. The equivalent of $g\in M_\beta^\star(p')$ is $g(w)|w|^{\frac{1}{q}-1}e^{-\frac{\beta}{2}|w|^{\frac{2}{q}}}\in L^{p'}(\mathbb{C},dA)$. Consider the following integral for $h\in L^{p'}(\mathbb{C},dA)$

$$\psi(h)=\int_{\mathbb{C}} h(w)\overline{g(w)}|w|^{\frac{2}{q}-2}e^{-\frac{\beta}{2}|w|^{\frac{2}{q}}}dA(w).$$

It suffices to show that ψ defined above is bounded functional on $L^p(\mathbb{C},dA)$. Without loss of generality, we assume that h has compact support in $\mathbb{C}$. In this case, the integral given as follows:

$$\int_{\mathbb{C}} h(w) K_q^{[\gamma]}(z,w) |w|^{\frac{2}{q}-2} e^{-\frac{\beta}{2}|w|^{\frac{2}{q}}} dA(w),$$

converges in the norm topology of $M_\alpha^\star(p)$ and thus

$$\begin{aligned}
\psi(h) &= \int_{\mathbb{C}} h(w) F_z\left(K_q^{[\gamma]}(z,w)\right) |w|^{\frac{2}{q}-2} e^{-\frac{\beta}{2}|w|^{\frac{2}{q}}} dA(w), \\
&= F\left(\int_{\mathbb{C}} h(w) K_q^{[\gamma]}(z,w) |w|^{\frac{2}{q}-2} e^{-\frac{\beta}{2}|w|^{\frac{2}{q}}} dA(w)\right), \\
&= \frac{\alpha}{\beta} F\left(\int_{\mathbb{C}} h(w) K_q^{[\alpha]}(z,w) |w|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|w|^{\frac{2}{q}}} dA(w)\right), \\
&= \frac{\pi q}{\beta}\frac{\alpha}{\pi q} F\left(\int_{\mathbb{C}} h\left(\left(\frac{\alpha}{\beta}\right)^{q/2} w\right) K_q^{[\alpha]}(z,w) |w|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|w|^{\frac{2}{q}}} dA(w)\right), \\
&= \frac{\pi q}{\beta} F\left(\frac{\alpha}{\pi q}\int_{\mathbb{C}} h\left(\left(\frac{\alpha}{\beta}\right)^{q/2} w\right) K_q^{[\alpha]}(z,w) |w|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|w|^{\frac{2}{q}}} dA(w)\right), \\
&= \frac{\pi q}{\beta} F\left(P_\alpha(\phi)\right),
\end{aligned}$$

where $\phi(z) = h\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right) |z|^{\frac{1}{q}-1} e^{\frac{\alpha}{2}|z|^{\frac{2}{q}}}$, since $h \in L^{p'}(\mathbb{C}, dA)$ which implies that $\phi \in L^p\left(\mathbb{C}, |z|^{1-\frac{1}{q}} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} dA(z\right.$ and since the projection P_α maps $L^p\left(\mathbb{C}, |z|^{\frac{1}{q}-1} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} dA(z)\right)$ boundedly onto $M_\alpha^\star(p)$ and therefore, we conclude that

$$|\psi(h)| \leq \frac{\pi q}{\beta} \|F\| \|P_\alpha(\phi)\|_{M_\alpha^\star(p)} \leq C\|h\|.$$

With this we have completed the proof of $g \in M_\beta^\star(p')$. Now, lastly, let $f = K_q^{[\gamma]}(z,a)$ for some $a \in \mathbb{C}$ then

$$\langle f, g\rangle_\gamma = \frac{\gamma}{\pi q} \lim_{R\to\infty} \int_{|z|<R} f(z)\overline{g(z)} |z|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} dA(z) = \overline{g(a)} = F(f).$$

Thus, whenever f is linear combination of kernel functions we have $F(f) = \langle f, g\rangle_\gamma$, as desired. $\square$

The relation given in (5.33) assumes that both f and g are linear combinations of kernel. Also, (5.34) ensures that the right-hand side of (5.33) converges for all $f \in M_{\alpha}^{\star}(p)$ and $g \in M_{\beta}^{\star}(p')$ and the integral is dominated by $\|f\|_{M_{\alpha}^{\star}(p)}\|g\|_{M_{\beta}^{\star}(p')}$. With all these proceeding and under the influence of LEMMA 5.16, we have following:

$$\lim_{R\to\infty}\int_{|z|<R} f(z)\overline{g(z)}|z|^{\frac{2}{q}-2}e^{-\gamma|z|^{\frac{2}{q}}}dA(z) = \int_{\mathbb{C}} f(z)\overline{g(z)}|z|^{\frac{2}{q}-2}e^{-\gamma|z|^{\frac{2}{q}}}dA(z),$$

and hence,

$$\lim_{R\to\infty}\int_{|z|<R} f(z)\overline{g(z)}d\mathbf{J}_{\gamma}(z) = \int_{\mathbb{C}} f(z)\overline{g(z)}d\mathbf{J}_{\gamma}(z).$$

Theorem 5.31. *Let $0 < p \leq 1$ and $\beta > 0$, then the dual space of $M_{\alpha}^{\star}(p)$ can be identified with $M_{\beta}^{\star}(\infty)$ under the integral pairing given as follows:*

$$\langle f, g\rangle_{\gamma} = \frac{\gamma}{\pi q}\lim_{R\to\infty}\int_{|z|<R} f(z)\overline{g(z)}|z|^{\frac{2}{q}-2}e^{-\gamma|z|^{\frac{2}{q}}}dA(z),$$

where $\gamma = (\alpha\beta)^{\frac{1}{2}}$.

Proof. Assume that $g \in M_{\beta}^{\star}(\infty)$ and as of now, we will show that $F(f)$ is a bounded linear functional over $M_{\alpha}^{\star}(p)$ for $0 < p \leq 1$ as follows:

$$F(f) = \frac{\alpha}{\pi q}\int_{\mathbb{C}} f(z)\overline{\phi(z)}|z|^{\frac{2}{q}-2}e^{-\alpha|z|^{\frac{2}{q}}}dA(z) = \frac{\alpha}{\pi q}\int_{\mathbb{C}}\left[f(z)|z|^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}\right]\left[\overline{\phi(z)}|z|^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}\right]dA(z),$$

where $\phi(z) = g\left(\left(\frac{\alpha}{\beta}\right)^{\frac{q}{2}} z\right)$, then we see that

$$|F(f)| \leq C\|\phi\|_{M_{\beta}^{\star}(\infty)}\|f\|_{M_{\alpha}^{\star}(1)} \leq C\|\phi\|_{M_{\beta}^{\star}(\infty)}\|f\|_{M_{\alpha}^{\star}(p)}.$$

This shows that F is bounded linear functional over $M^\star_\alpha(p)$ and now we will show that $g \in M^\star_\beta(\infty)$ and for that we first consider $\overline{g(w)} = F_z\left(K^{[\gamma]}_q(z,w)\right)$. Then,

$$
\begin{aligned}
|g(w)|^p &\leq \frac{p\alpha}{2\pi q}\|F\|^p \int_{\mathbb{C}} \left|K^{[\gamma]}_q(z,w)\right|^p |z|^{\frac{p}{q}-p} e^{-\frac{p\alpha}{2}|z|^{\frac{2}{q}}} dA(z), \\
&= \frac{p\alpha}{2\pi q}\|F\|^p \int_{\mathbb{C}} \left[K^{[\gamma]}_q(z,w)\overline{K^{[\gamma]}_q(z,w)}\right]^{\frac{p}{2}} |z|^{\frac{p}{q}-p} e^{-\frac{p\alpha}{2}|z|^{\frac{2}{q}}} dA(z), \\
&= \frac{p\alpha}{2\pi q}\|F\|^p \int_{\mathbb{C}} \left[K^{[\gamma]}_q(z,w)\right]^{\frac{p}{2}} \left[K^{[\gamma]}_q(w,z)\right]^{\frac{p}{2}} |z|^{\frac{p}{q}-p} e^{-\frac{p\alpha}{2}|z|^{\frac{2}{q}}} dA(z), \\
&= \|F\|^p \left[K^{[\gamma]}_q(w,w)\right]^{\frac{p}{2}}.
\end{aligned}
$$

This shows that $g \in M^\star_\beta(\infty)$ with $\|g\|_{M^\star_\beta(\infty)} \leq \|F\|$. ☐

Corollary 5.32. *1. Let $1 \leq p < \infty$, then the dual of $M^\star_\alpha(p)$ can be identified by $M^\star_\alpha(p')$ where $\frac{1}{p'} + \frac{1}{p} = 1$.*

2. If $0 < p < 1$, then the dual of $M^\star_\alpha(p)$ can be identified by $M^\star_\alpha(\infty)$ under the same integral pairing $\langle f, g\rangle_\gamma$.

Theorem 5.33. *The dual of $f^\infty_{M^\star_\alpha}$ can be identified by $M^\star_\beta(1)$ under the integral pairing of $\langle f, g\rangle_\gamma$.*

Proof. If $g \in M^\star_\beta(1)$, then by the Theorem 5.31, we have $F(f) = \langle f, g\rangle_\gamma$ defines a bounded linear functional on $f^\infty_{M^\star_\alpha}$. With this F in our hand, we have the set of linear combination of kernels to be dense in $f^\infty_{M^\star_\alpha}$ not in $M^\star_\alpha(\infty)$, we have following:

$$
F(f) = \frac{\gamma}{\pi q} \lim_{R\to\infty} \int_{|w|<R} f(w)\overline{g(w)}|w|^{\frac{2}{q}-2} e^{-\gamma|w|^{\frac{2}{q}}} dA(w),
$$

for f in a dense subset of $f^\infty_{M^\star_\alpha}$, where $\overline{g(w)} = F_z\left(K^{[\gamma]}_q(z,w)\right)$. Now, we will show that $g \in M^\star_\beta(1)$ by showing following:

$$
|\langle f, g\rangle_\gamma| \leq C\|f\|_{M^\star_\alpha(\infty)},
$$

for all $f \in M_\alpha^\star(\infty)$. This is so because, since the dual space of $M_\beta^\star(1)$ is identified with $M_\alpha^\star(\infty)$ under the integral pairing $\langle f, g\rangle_\gamma$. Let

$$f_n(z) = f\left(\frac{n}{n+1}z\right),$$

$\forall n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$. It is clear that $f \in M_\alpha^\star(\infty)$ and $\|f_n\|_{M_\alpha^\star(\infty)} \leq \|f\|_{M_\alpha^\star(\infty)}$, then following chain of equalities follows:

$$\begin{aligned}
\langle f, g\rangle_\gamma &= \frac{\gamma}{\pi q} \lim_{R\to\infty} \int_{|w|<R} f(w) F_z\left(K_q^{[\gamma]}(z,w)\right) |w|^{\frac{2}{q}-2} e^{\gamma|w|^{\frac{2}{q}}} dA(w), \\
&= F\left(\frac{\gamma}{\pi q} \lim_{R\to\infty} \int_{|w|<R} f(w) K_q^{[\gamma]}(z,w) |w|^{\frac{2}{q}-2} e^{-\gamma|w|^{\frac{2}{q}}} dA(w)\right), \\
&= \lim_{n\to\infty} F\left(\frac{\gamma}{\pi q} \lim_{R\to\infty} \int_{|w|<R} f_n(w) K_q^{[\gamma]}(z,w) |w|^{\frac{2}{q}-2} e^{-\gamma|w|^{\frac{2}{q}}} dA(w)\right), \\
&= \lim_{n\to\infty} F\left(\frac{\gamma}{\pi q} \int_{\mathbb{C}} f_n(w) K_q^{[\gamma]}(z,w) |w|^{\frac{2}{q}-2} e^{-\gamma|w|^{\frac{2}{q}}} dA(w)\right), \\
&= \lim_{n\to\infty} F(f_n).
\end{aligned}$$

With the above observations, following can be deduced:

$$|\langle f, g\rangle_\gamma| \leq \|F\| \|f_n\|_{M_\alpha^\star(\infty)} \leq \|F\| \|f\|_{M_\alpha^\star(\infty)},$$

for all $f \in M_\alpha^\star(\infty)$. This implies that $g \in M_\beta^\star(1)$ and we are done. □

5.4 Complex Interpolation

Theorem 5.34. *Suppose w, w_0, and w_1 are positive weight functions on $\mathbb{C}$ and if $1 \leq p_0 \leq p_1 \leq \infty$ and $\theta \in [0,1]$, then*

$$[L^{p_0}(\mathbb{C}, w_0 dA), L^{p_1}(\mathbb{C}, w_1 dA)]_\theta = L^p(\mathbb{C}, w dA),$$

with equal norms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad w^{\frac{1}{p}} = w_0^{\frac{1-\theta}{p_0}} w_1^{\frac{\theta}{p_1}}.$$

Proof. Follow [47] or [10] for this. $\square$

L^p_α contains all the Lebesgue measurable function on $\mathbb{C}$ such that $f(z)|z|^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} \in L^p(\mathbb{C}, dA)$. With the inherited norm, $M^\star_\alpha(p)$ is the closed subspace of L^p_α consisting of entire functions.

Corollary 5.35. *Suppose* $1 \leq p_0 \leq p_1 \leq \infty$ *and* $\theta \in [0,1]$*, then for any positive weight parameters* α_0 *and* α_1*, we have following*

$$\left[L^{p_0}_{\alpha_0}, L^{p_1}_{\alpha_1}\right]_\theta = L^p_\alpha,$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \alpha = \alpha_0(1-\theta) + \alpha_1\theta.$$

Proof. It follows from the Stein–Weiss interpolation theorem that $\left[L^{p_0}_{\alpha_0}, L^{p_1}_{\alpha_1}\right]_\theta$ is

$$\begin{aligned}
&= \left[L^{p_0}\left(\mathbb{C}, |z|^{\frac{1}{q}-1}e^{-\frac{\alpha_0}{2}|z|^{\frac{2}{q}}}dA(z)\right), L^{p_0}\left(\mathbb{C}, |z|^{\frac{1}{q}-1}e^{-\frac{\alpha_1}{2}|z|^{\frac{2}{q}}}dA(z)\right)\right]_\theta, \\
&= L^p\left(\mathbb{C}, |z|^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}dA(z)\right) = L^p_\alpha.
\end{aligned}$$

This proves the desired result under the observation of $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$ and $\alpha = \alpha_0(1-\theta) + \alpha_1\theta$. $\square$

We consider the case when the weight parameter α is fixed.

Theorem 5.36. *With* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $1 \leq p_0 \leq p_1 \leq \infty$ *and* $\theta \in [0,1]$*, then* $[M^\star_\alpha(p_0), M^\star_\alpha(p_1)]_\theta = M^\star_\alpha(p)$.

Proof. The inclusion of $[M^\star_\alpha(p_0), M^\star_\alpha(p_1)]_\theta$ in $M^\star_\alpha(p)$ follows from the definition of complex interpolation, the fact that each $M^\star_\alpha(p_k)$ is a closed subspace of $L^{p_k}_\alpha$ and we already have the interpolation done on $L^{p_k}_\alpha$ in COROLLARY 5.35. Now, we proceed further by showing the other side of the inclusion by picking $f \in M^\star_\alpha(p) \subset L^p_\alpha$ which is entire. Due to the interpolation there exist a function $F(z, \zeta)$ with $z \in \mathbb{C}$ and $0 \leq \text{Re}(\zeta) \leq 1$ with following conditions, where $\delta = 0$ and 1 and $\tilde{C}$ is a positive constant.

1. $F(z, \theta) = f(z)$ for all $z \in \mathbb{C}$
2. $\|F(\cdot, \zeta)\|_{M^\star_\alpha(p_\delta)} \leq \tilde{C}$ with $\text{Re}(\zeta) = \delta$.

Define $G(z, \zeta)$ as follows

$$G(z, \zeta) = \frac{\alpha}{\pi q} \int_{\mathbb{C}} K^{[\alpha]}_q(z, w) F(w, \zeta) |w|^{\frac{2}{q} - 2} e^{-\alpha |w|^{\frac{2}{q}}} dA(w).$$

By the COROLLARY 5.29, we have following

1. $G(z, \theta) = f(z)$ for all $z \in \mathbb{C}$
2. $\|G(\cdot, \zeta)\|_{M^\star_\alpha(p_\delta)} \leq C \left[\left(\dfrac{\alpha}{\alpha - p'\delta} \right)^{\frac{1}{p'}} \left(\dfrac{\alpha}{\beta - p\delta} \right)^{\frac{1}{p'}} \right] \tilde{C}$ with $\text{Re}(\zeta) = \delta$.

As $z \to G(z, \zeta)$ is entire thus, this implies $f \in M^\star_\alpha(p) \subset [M^\star_\alpha(p_0), M^\star_\alpha(p_1)]_\theta$, and thus our desired result is achieved. □

Now, we consider the case when we have different weight parameters α_0 and α_1 as follows.

Theorem 5.37. *Suppose* $1 \leq p_0 \leq p_1 \leq \infty$ *and* $\theta \in [0, 1]$*, then for any positive weight parameters* α_0 *and* α_1*, we have*

$$\left[M^\star_{\alpha_0}(p_0), M^\star_{\alpha_1}(p_1) \right]_\theta = M^\star_\alpha(p),$$

for

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \alpha = \alpha_0^{1-\theta} \alpha_1^\theta.$$

Proof. Consider a dilation operator ζ defined as follows:

$$S_\zeta f(z) = \zeta_q^{1-\frac{2}{p}} f\left(\left(\frac{\alpha_0}{\alpha_1}\right)^{(\zeta-\theta)\frac{q}{2}} z\right),$$

where $\zeta_q = \zeta^{\frac{1}{q}-1}$. By the THEOREM 5.10 we have S_ζ an isometry from $M^\star_{\alpha_0}(p_0)$ to $M^\star_\alpha(p_0)$ when $\mathrm{Re}(\zeta) = 0$. On the other hand, S_ζ is again an isometry from $M^\star_{\alpha_0}(p_1)$ to $M^\star_\alpha(p)$ when $\mathrm{Re}(\zeta) = 1$. Furthermore, both S_ζ and its inverse are analytic in ζ whenever f is so. Thus by the abstract Stein interpolation theorem, we have the operator S_θ must be an isometry from $\left[M^\star_{\alpha_0}(p_0), M^\star_{\alpha_1}(p_1)\right]_\theta$ to $[M^\star_\alpha(p_0), M^\star_\alpha(p_1)]_\theta$. Since, $S_\theta = I$, is the identity operator, we must have

$$[M^\star_\alpha(p_0), M^\star_\alpha(p_1)]_\theta = M^\star_\alpha(p),$$

where this step follows from THEOREM 5.36. □

5.5 Atomic Decomposition

This section provides the atomic decomposition of $M^\star_\alpha$ which is of high importance in applied direction.

Proposition 5.38. *Suppose α and $p > 0$. Let K be a non-empty, closed and bounded subset of $\mathbb{C}$ and for $s \in [0, r]$ define $\mathcal{E}_{\alpha,p,q}(s|a) : K \times [0, r] \to \mathbb{R}_+$ by following:*

$$\mathcal{E}_{\alpha,p,q}(s|a) := e^{-\frac{\alpha p}{2}\left[|s+a|^{\frac{2}{q}} - \left(|s|^2+|a|^2\right)^{\frac{1}{q}}\right]}. \tag{5.35}$$

Then $\mathcal{E}_{\alpha,p,q}(s|a)$ attains both maximum and minimum over $K \times [0, r]$ for $s \in [0, r]$.

Proof. To begin with, we note that $\mathcal{E}_{\alpha,p,q}(s|a)$ is continuous and also we have, by the closed and bounded nature of $K \times [0, r]$ which makes it compact. Combining these two, we have the image of $K \times [0, r]$ to be compact in $\mathbb{R}_+$ under the mapping $\mathcal{E}_{\alpha,p,q}(s|a)$ and therefore, since every compact subset of $\mathbb{R}_+$ has both maximum and minimum values, our desired result follows immediately. $\square$

Let $m_{\mathcal{E}_{\alpha,p,q}(s|a)}$ be the minimum of $\mathcal{E}_{\alpha,p,q}(s|a)$ and will be used in the following lemma.

Lemma 5.39. *For any positive parameters α, p, and R, we have following:*

$$\left|f(a)a^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|a|^{\frac{2}{q}}}\right|^p \leq \frac{1}{r^2 m_{\mathcal{E}_{\alpha,p,q}(s|a)}}\int_{B(a,r)}\left|f(z)z^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}\right|^p dA(z),$$

for all entire functions f, all complex numbers a, and all $r \in (0,R]$ and $m_{\mathcal{E}_{\alpha,p,q}(s|a)}$ is coming from the Proposition *5.38 for $s \in [0,r]$.*

Proof. To begin with, we first consider following equalities:

$$\begin{aligned}|w+a|^{\frac{2}{q}} = \left[|w|^2+w\overline{a}+\overline{w}a+|a|^2\right]^{\frac{1}{q}} &= \sum_{k=0}^{\infty}\binom{1/q}{k}\left(|w|^2+|a|^2\right)^{\frac{1}{q}-k}\left(w\overline{a}+\overline{w}a\right)^k, \\ &= \left(|w|^2+|a|^2\right)^{\frac{1}{q}} + \sum_{k=1}^{\infty}\binom{\frac{1}{q}}{k}\left(|w|^2+|a|^2\right)^{\frac{1}{q}-k}\left(w\overline{a}+\overline{w}a\right)^k.\end{aligned}$$

This imply that we have following:

$$e^{-\frac{\alpha p}{2}|w+a|^{\frac{2}{q}}} = e^{-\frac{\alpha p}{2}\left(|w|^2+|a|^2\right)^{\frac{1}{q}}}e^{-\frac{\alpha p}{2}\sum_{k=1}^{\infty}\binom{\frac{1}{q}}{k}\left(|w|^2+|a|^2\right)^{\frac{1}{q}-k}\left(w\overline{a}+\overline{w}a\right)^k}.$$

Let $F(w,a) = f(w+a)(w+a)^{\frac{1}{q}-1}e^{-\frac{\alpha p}{2}\left(|w|^2+|a|^2\right)^{\frac{1}{q}}}$. We consider the integral on the right side of above lemma and then following follows:

$$I = \int_{B(a,r)}\left|f(z)z^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}\right|^p dA(z),$$

and then following consequences follows up:

$$
\begin{aligned}
I &= \int_{|w|<r} \left| f(w+a)(w+a)^{\frac{1}{q}-1} e^{-\frac{\alpha}{2}|w+a|^{\frac{2}{q}}} \right|^p dA(w),\\
&= \int_{|w|<r} \left| f(w+a)(w+a)^{\frac{1}{q}-1} \right|^p e^{-\frac{\alpha p}{2}|w+a|^{\frac{2}{q}}} dA(w),\\
&= \int_{|w|<r} |F(w,a)|^p \, e^{-\frac{\alpha p}{2}\sum_{k=1}^{\infty}\binom{\frac{1}{q}}{k}\left(|w|^2+|a|^2\right)^{\frac{1}{q}-k}(w\bar{a}+\overline{w}a)^k} dA(w),\\
&\geq |F(0,a)|^p \int_{|w|<r} e^{-\frac{\alpha p}{2}\sum_{k=1}^{\infty}\binom{\frac{1}{q}}{k}\left(|w|^2+|a|^2\right)^{\frac{1}{q}-k}(w\bar{a}+\overline{w}a)^k} dA(w),\\
&= |f(a)|^p |a|^{\frac{p}{q}-p} e^{-\frac{\alpha p}{2}|a|^{\frac{2}{q}}} \int_{|w|<r} e^{-\frac{\alpha p}{2}\sum_{k=1}^{\infty}\binom{\frac{1}{q}}{k}\left(|w|^2+|a|^2\right)^{\frac{1}{q}-k}(w\bar{a}+\overline{w}a)^k} dA(w),\\
&= |f(a)|^p |a|^{\frac{p}{q}-p} e^{-\frac{\alpha p}{2}|a|^{\frac{2}{q}}} \int_{|w|<r} e^{-\frac{\alpha p}{2}\left[|w+a|^{\frac{2}{q}}-\left(|w|^2+|a|^2\right)^{\frac{1}{q}}\right]} dA(w),\\
&= 2\pi |f(a)|^p |a|^{\frac{p}{q}-p} e^{-\frac{\alpha p}{2}|a|^{\frac{2}{q}}} \int_0^r e^{-\frac{\alpha p}{2}\left[|s+a|^{\frac{2}{q}}-\left(|s|^2+|a|^2\right)^{\frac{1}{q}}\right]} s ds,\\
&\leq 2\pi |f(a)|^p |a|^{\frac{p}{q}-p} e^{-\frac{\alpha p}{2}|a|^{\frac{2}{q}}} m_{\mathcal{E}_{\alpha,p,q}(s|a)} \frac{r^2}{2},\\
&= \pi |f(a)|^p |a|^{\frac{p}{q}-p} e^{-\frac{\alpha p}{2}|a|^{\frac{2}{q}}} m_{\mathcal{E}_{\alpha,p,q}(s|a)} r^2.
\end{aligned}
$$

Thus, our desired result is achieved and from now-on-wards $m_{\mathcal{E}_{\alpha,p,q}}$ will be used to represent $m_{\mathcal{E}_{\alpha,p,q}(s|a)}$. □

Lemma 5.40. *Let R_{00} be the fundamental region Λ where Λ is same as given in Section 1.2 in [53], then for any positive number δ, following is true for all $w \in \mathbb{C}$:*

$$
\sum_{z\in\Lambda} e^{-\delta\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2} \leq C.
$$

Proof. We will proof the above result for the fundamental region R_{00} of Λ by realizing the fact that $\left|\frac{w}{z}\right| \leq \frac{1}{2}$, which is why

$$
\left||z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right|^2 = |z|^{\frac{2}{q}}\left|1-\left|\frac{w}{z}\right|^{\frac{1}{q}}\right|^2 \geq |z|^{\frac{2}{q}}\left|1-\left|\frac{1}{2}\right|^{\frac{1}{q}}\right|^2.
$$

Since, $\sum_{z\in\Lambda} e^{-\delta|z|^{\frac{2}{q}}\left[1-\left|\frac{1}{2}\right|^{\frac{1}{q}}\right]^2}$ is obviously convergent, we obtain the desired result. □

Proposition 5.41. *We propose following as an* asymptotically equivalence *relation for a fixed* $z \in \mathbb{C}$

$$\left|k_u^{[\alpha]}(z) - k_w^{[\alpha]}(z)\right| \approx C \left| \frac{e^{\alpha|zu|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}}} - \frac{e^{\alpha|zw|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}}} \right|, \tag{5.36}$$

under the situation $|u-w| < \delta$*, where* δ *is arbitrarily positive.*

Proof. To understand above, we first consider the right side of above and proceed as follows:

$$\begin{aligned}
\left| \frac{e^{\alpha|zu|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}}} - \frac{e^{\alpha|zw|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}}} \right| &= \frac{1}{e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}} e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}}} \left| e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}} e^{\alpha|zu|^{\frac{1}{q}}} - e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}} e^{\alpha|zw|^{\frac{1}{q}}} \right|, \\
&= \frac{1}{e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}} e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}}} \left| \sum_{n,m\geq 0} \frac{\alpha^{n+m}}{n!\,m!} \frac{|zu|^{\frac{n}{q}}|w|^{\frac{2m}{q}}}{2^m} - \sum_{k,l\geq 0} \frac{\alpha^{k+l}}{k!\,l!} \frac{|zu|^{\frac{k}{q}}|w|^{\frac{2l}{q}}}{2^l} \right|, \\
&= \frac{1}{e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}} e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}}} \left| \sum_{n,m\geq 0} \sum_{k,l\geq 0} \left[\frac{\alpha^{n+m}}{n!\,m!} \frac{|zu|^{\frac{n}{q}}|w|^{\frac{2m}{q}}}{2^m} - \frac{\alpha^{k+l}}{k!\,l!} \frac{|zu|^{\frac{k}{q}}|w|^{\frac{2l}{q}}}{2^l} \right] \right|, \\
&\approx \frac{1}{e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}} e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}}} \left| \sum_{n,m\geq 0} \frac{\alpha^{n+m}}{2^m n!\,m!} |zu|^{\frac{n}{q}}|w|^{\frac{2m}{q}} - \sum_{k,l\geq 0} \frac{\alpha^{k+l}}{2^l k!\,l!} |zu|^{\frac{k}{q}}|w|^{\frac{2l}{q}} \right|.
\end{aligned}$$

Now, fixing z and grouping all the terms present above at the same index makes the result immediately 0, and therefore we see that the difference is asymptotically equivalent to $|z|^{j_1}|u-w|^{j_2}$ for some $j_1, \& j_2 \in \mathbb{Z}_+$ on unequal indices. Before proceeding further we first recall two basic inequalities for $E_q(\alpha^q|u|^2)$ and $E_q(\alpha^q|u|^2)$ as follows:

$$C_{1\bullet w} e^{\alpha|w|^{\frac{2}{q}}} \leq E_q(\alpha^q|w|^2) \leq C_{2\bullet w} e^{\alpha|w|^{\frac{2}{q}}}, \tag{5.37}$$

and

$$C_{1\bullet u} e^{\alpha|u|^{\frac{2}{q}}} \leq E_q(\alpha^q|u|^2) \leq C_{2\bullet u} e^{\alpha|u|^{\frac{2}{q}}}. \tag{5.38}$$

Now, we have following:

$$
\begin{aligned}
k_w^{[\alpha]}(z) &= \frac{K_q^{[\alpha]}(z,w)}{\sqrt{K_q^{[\alpha]}(w,w)}}, \\
&= \frac{K_q^{[\alpha]}(z,w)}{\sqrt{E_q(\alpha^q |w|^2)}}, \\
&\geq \frac{K_q^{[\alpha]}(z,w)}{C_{2\bullet w} e^{\alpha |w|^{\frac{2}{q}}}}, \\
\implies \left| k_u^{[\alpha]}(z) - k_w^{[\alpha]}(z) \right| &\leq \left| \frac{K_q^{[\alpha]}(z,u)}{C_{1\bullet u} e^{\alpha |u|^{\frac{2}{q}}}} - \frac{K_q^{[\alpha]}(z,w)}{C_{2\bullet w} e^{\alpha |w|^{\frac{2}{q}}}} \right|.
\end{aligned}
$$

The above quantity is same as to have following:

$$
\frac{1}{C_{1\bullet u} e^{\alpha |u|^{\frac{2}{q}}} C_{2\bullet w} e^{\alpha |w|^{\frac{2}{q}}}} \left| C_{2\bullet w} \sum_{n\geq 0} \frac{\left(\frac{\alpha}{2}|w|^{\frac{2}{q}}\right)^n}{n!} \sum_{m\geq 0} \frac{(\alpha^q z\overline{u})^n}{\Gamma(qm+1)} - C_{1\bullet u} \sum_{k\geq 0} \frac{\left(\frac{\alpha}{2}|w|^{\frac{2}{q}}\right)^k}{k!} \sum_{l\geq 0} \frac{(\alpha^q z\overline{u})^l}{\Gamma(ql+1)} \right|.
$$

Again, with the same reasoning of *asymptotically equivalence* as given above, here we have $z^{j_1}|u-w|^{j_2}$ for some $j_1, \& j_2 \in \mathbb{Z}_+$ on unequal indices. Combining these two above results and with the fact that $|u-w| < \delta$, thus (5.36) is proved. □

Proposition 5.42. *For $C > 0$, we have following:*

$$
\left| k_u^{[\alpha]}(z) + k_w^{[\alpha]}(z) \right| \leq C \left| \frac{e^{\alpha |zu|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}}} + \frac{e^{\alpha |zw|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}}} \right|. \tag{5.39}
$$

Proof. Consider the left side of above and then

$$
\begin{aligned}
\left| k_u^{[\alpha]}(z) + k_w^{[\alpha]}(z) \right| &\leq \left| k_u^{[\alpha]}(z) \right| + \left| k_w^{[\alpha]}(z) \right|, \\
&\leq \frac{|K_q^{[\alpha]}(z,u)|}{C_{1\bullet u} e^{\alpha |u|^{\frac{2}{q}}}} + \frac{|K_q^{[\alpha]}(z,w)|}{C_{1\bullet w} e^{\alpha |w|^{\frac{2}{q}}}}, \\
&= \frac{|E_q(\alpha^q z\overline{u})|}{C_{1\bullet w} e^{\alpha |w|^{\frac{2}{q}}}} + \frac{|E_q(\alpha^q z\overline{w})|}{C_{1\bullet w} e^{\alpha |w|^{\frac{2}{q}}}},
\end{aligned}
$$

$$\leq \frac{C_{1\bullet z,u} e^{\alpha|zu|^{\frac{1}{q}}}}{C_{1\bullet w} e^{\alpha|w|^{\frac{2}{q}}}} + \frac{C_{1\bullet z,w} e^{\alpha|zw|^{\frac{1}{q}}}}{C_{1\bullet w} e^{\alpha|w|^{\frac{2}{q}}}},$$

$$\approx C \left| \frac{e^{\alpha|zu|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}}} + \frac{e^{\alpha|zw|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}}} \right|.$$

□

Recall that for any positive number r,

$$r\mathbb{Z}^2 \coloneqq \{r(m+in) : m, n \in \mathbb{Z}^2\},$$

is a square lattice in $\mathbb{C}$, with following as the *fundamental region* which is obtained by removing the boundary points

$$S_r = \left\{ z = x + iy : x, y \in \left[-\frac{r}{2}, \frac{r}{2} \right) \right\}.$$

The following decomposition of $\mathbb{C}$ is admissible:

$$\mathbb{C} = \bigcup \left\{ S_r + z : z \in r\mathbb{Z}^2 \right\},$$

and therefore,

$$\int_{\mathbb{C}} f(z) = \sum_{w \in r\mathbb{Z}^2} \int_{S_r + w} f(z) d\mu(z),$$

since, the fundamental region defined by S_r doesn't overlap with each-other; for $f \in L^1(\mathbb{C}, d\mu)$.

Theorem 5.43. *We may have following* $0 < p \leq \infty$ *with*

$$f(z) = \sum_{w \in r\mathbb{Z}^2} c_w k_w^{[\alpha]}(z),$$

where $c_w \in \ell^p$ for $w \in r\mathbb{Z}^2$. Moreover, there exists a positive constant C (independent of f) such that

$$\frac{1}{C}\|f\|_{M_\alpha^\star(p)} \leq \inf\{\|c_w\|\}_{\ell^p} \leq C\|f\|_{M_\alpha^\star(p)},$$

for all $f \in M_\alpha^\star(p)$.

Proof. To begin with, assume that $0 < p \leq 1$ and $f(z) = \sum_{w \in r\mathbb{Z}^2} c_w k_w^{[\alpha]}(z)$, with $\{c_w\} \in \ell^p$. Then by Holder's inequality and $k_w^{[\alpha]}(z)$ unit vector in $M_\alpha^\star(p)$

$$\left|f(z) z^{\frac{1}{q}-1} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}\right|^p \leq \sum_{w \in r\mathbb{Z}^2} |c_w|^p \left|k_w^{[\alpha]}(z) z^{\frac{1}{q}-1} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}\right|^p,$$

$$\|f\|_{M_\alpha^\star(p)}^p \leq \sum_{w \in r\mathbb{Z}^2} |c_w|^p.$$

Thus, $f \in M_\alpha^\star(p)$ and $\|f\|_{M_\alpha^\star(p)}^p \leq \inf \sum_{w \in r\mathbb{Z}^2} |c_w|^p$. Now, $\{c_w\} \in \ell^\infty$, then

$$\begin{aligned}
|f(z)| e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} &\leq C' \|\{c_w\}\|_\infty \sum_{w \in r\mathbb{Z}^2} |k_w^{[\alpha]}(z)| e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}, \\
&\leq C' \|\{c_w\}\|_\infty \sum_{w \in r\mathbb{Z}^2} \frac{C e^{\alpha|zw|^{\frac{1}{q}}}}{\left(C_1 e^{\alpha|w|^{\frac{2}{q}}}\right)^{\frac{1}{2}}} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}, \\
&\leq \tilde{C} \|\{c_w\}\|_\infty \sum_{w \in r\mathbb{Z}^2} e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}} - |w|^{\frac{1}{q}}\right]^2}, \\
&\leq C \|\{c_w\}\|_\infty \qquad \text{(\textit{by} LEMMA 5.40, \textit{other} } C > 0).
\end{aligned}$$

Taking the infimum is taken over all sequences $\{c_w\}_{\ell^\infty}$ to have $\|f\|_{M_\alpha^\star(\alpha)} \leq C \inf \|\{c_w\}\|_{\ell^\infty}$. After interpolating between $p = 1$ and $p = \infty$, we have now shown that, for all $0 < p \leq \infty$ and $f(z) = \sum_{w \in r\mathbb{Z}^2} c_w k_w^{[\alpha]}(z)$, is in $M_\alpha^\star(p)$ $\{c_w\} \in \ell^p$ and finally, $\|f\|_{M_\alpha^\star(p)} \leq C \inf \|\{c_w\}\|_{\ell^p}$, where C is positive constant depending on α, p and r and the infimum is taken over all sequences $\{c_w\} \in \ell^p$, that give rise to the representation of f as $f(z) = \sum_{w \in r\mathbb{Z}^2} c_w k_w^{[\alpha]}(z)$. To prove the other part of the theorem, we assume that $0 < r < 1$ and consider the linear operator T_r defined on the space of entire functions as follows:

$$T_r f(z) = \frac{\alpha}{\pi q} \sum_{w \in r\mathbb{Z}^2} k_w^{[\alpha]}(z) \int_{S_r+w} f(u)|u|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}} e^{-\alpha i \operatorname{Im}(w\overline{u})} dA(u).$$

We claim that $\|I - T_r\| < 1$ on $M_\alpha^\star(p)$, where I is the identity operator, which is why T_r is a bounded linear operator on $M_\alpha^\star(p)$. We proceed as follows by realizing that

$$\begin{aligned} f(z) &= \int_{\mathbb{C}} f(u) K_q^{[\alpha]}(z,u) d\mathbf{J}_\alpha(u), \\ &= \frac{\alpha}{\pi q} \sum_{w \in r\mathbb{Z}^2} \int_{S_r+w} f(u) K_q^{[\alpha]}(z,u)|u|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}} - \alpha i \operatorname{Im}(w\overline{u})} e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}} + \alpha i \operatorname{Im}(w\overline{u})} dA(u). \end{aligned}$$

It follows that $D_r = I - T_r$ is as follows:

$$D_r f(z) = \frac{\alpha}{\pi q} \sum_{w \in r\mathbb{Z}^2} \int_{S_r+w} f(u) H(z,w,u) dA(u), \tag{5.40}$$

where $H(z,w,u) = \left[k_w^{[\alpha]}(z) - k_u^{[\alpha]}(z) e^{-\alpha i \operatorname{Im}(w\overline{u})}\right] |u|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}} + \alpha i \operatorname{Im}(w\overline{u})}$. We now estimate the norm of the operator D_r on $M_\alpha^\star(\infty)$ and on $M_\alpha^\star(1)$.

Case 1: $p = \infty$ By (5.40) we have $|D_r f| e^{-\dfrac{\alpha}{2}|z|^{\frac{2}{q}}} \leq \frac{\alpha}{\pi q} \|f\|_{M_\alpha^\star(\infty)} J_r(z)$, and

$$\begin{aligned} J_r(z) &= \sum_{w \in r\mathbb{Z}^2} \int_{S_r+w} \left| k_u^{[\alpha]}(z) e^{-\alpha i \operatorname{Im}(w\overline{u})} - k_w^{[\alpha]}(z) \right| |z|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}} + \alpha i \operatorname{Im}(w\overline{u})} dA(u), \\ &\leq \sum_{w \in r\mathbb{Z}^2} \int_{S_r+w} \left[\left| k_u^{[\alpha]}(z) \right| + \left| k_w^{[\alpha]}(z) \right| \right] |z|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}} + \alpha i \operatorname{Im}(w\overline{u})} dA(u), \\ &\leq \sum_{w \in r\mathbb{Z}^2} \int_{S_r+w} \left[\left| k_u^{[\alpha]}(z) - k_w^{[\alpha]}(z) \right| + \left| k_u^{[\alpha]}(z) + k_w^{[\alpha]}(z) \right| \right] \\ &\qquad |z|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}} + \alpha i \operatorname{Im}(w\overline{u})} dA(u), \\ &\leq \sum_{w \in r\mathbb{Z}^2} \int_{S_r+w} \left[C_1 \left| \frac{e^{\alpha|zu|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}}} - \frac{e^{\alpha|zw|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}}} \right| + C_2 \left| \frac{e^{\alpha|zu|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|u|^{\frac{2}{q}}}} + \frac{e^{\alpha|zw|^{\frac{1}{q}}}}{e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}}} \right| \right] \\ &\qquad |z|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}} + \alpha i \operatorname{Im}(w\overline{u})} dA(u), \end{aligned}$$

$$\approx\leq \sum_{w\in r\mathbb{Z}^2}\int_{S_r+w} C|z|^{\frac{2}{q}-2}\Bigg[\Bigg|e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|u|^{\frac{1}{q}}\right]^2}-e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2}\Bigg| + \Bigg|e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|u|^{\frac{1}{q}}\right]^2}+e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2}\Bigg|\Bigg]dA(u).$$

Therefore, $J_r(z)$ is less than or equal to following:

$$J_r(z) \leq \mathcal{E}(z,w), \tag{5.41}$$

where $\mathcal{E}(z,w)$ is given as follows.

$$\mathcal{E}(z,w)=\sum_{w\in r\mathbb{Z}^2}\int_{S_r+w} C|z|^{\frac{2}{q}-2}e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2}\left[\left|1-\frac{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|u|^{\frac{1}{q}}\right]^2}}{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2}}\right|+\left|1+\frac{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|u|^{\frac{1}{q}}\right]^2}}{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2}}\right|\right]dA(u).$$

Now, for any $\zeta\in\mathbb{C}$ we have $|1+e^{\zeta}|+|1-e^{\zeta}|\leq 1+e^{|\zeta|}+e^{|\zeta|}-1=2e^{|\zeta|}$, applying this to have following:

$$\begin{aligned}\left[\left|1-\frac{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|u|^{\frac{1}{q}}\right]^2}}{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2}}\right|+\left|1+\frac{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|u|^{\frac{1}{q}}\right]^2}}{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2}}\right|\right] &\leq 2e^{\frac{\alpha}{2}\left[\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2-\left[|z|^{\frac{1}{q}}-|u|^{\frac{1}{q}}\right]^2\right]},\\ &\leq 2e^{\frac{\alpha}{2}\left[\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2-\left[|z|^{\frac{1}{q}}-r^{\frac{1}{q}}\right]^2\right]},\\ &\leq 2re^{\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2}.\end{aligned}\tag{5.42}$$

Combining (5.41) and (5.42) to have following:

$$\begin{aligned}J_r(z) &\leq 2Cr|z|^{\frac{2}{q}-2}\sum_{w\in r\mathbb{Z}^2}e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2}e^{\frac{\alpha}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]^2}\int_{S_r+w}dA(u),\\ &= 2Cr^3\frac{1}{m_{\mathcal{E}_{\alpha,2,q}}r^2}, \qquad (m_{\mathcal{E}_{\alpha,p,q}} \textit{ in } \textsc{Lemma } 5.39).\end{aligned}$$

This shows that in conclusion, we have following, independent of $0 < r < 1$,

$$\|D_r f\|_{M^\star_\alpha(\infty)} \leq Cr\|f\|_{M^\star_\alpha(\infty)} \implies \|D_r\|_{M^\star_\alpha(\infty)} < 1.$$

To estimate the norm of D_r on $M^\star_\alpha(1)$, first note that $|D_r f(z)|$ is less than or equal to following:

$$\frac{\alpha}{\pi q} \sum_{w \in r\mathbb{Z}^2} \int_{S_r + w} \left| k_w^{[\alpha]}(z) - k_u^{[\alpha]}(z) e^{-\alpha i \operatorname{Im}(w\overline{u})} \right| |f(u)||u|^{\frac{1}{q}-1} e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}} dA(u).$$

By Fubini's theorem, the integral $\int_{\mathbb{C}} |D_r f(z)| e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} |z|^{\frac{1}{q}-1} dA(z)$ is less than or equal to

$$\frac{\alpha}{q\pi} \sum_{w \in r\mathbb{Z}^2} \int_{S_r + w} |f(u)||u|^{\frac{1}{q}-1} e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}} H(w,u) dA(u),$$

where $H(w,u)$ is defined as follows:

$$\begin{aligned} H(w,u) &= \int_{\mathbb{C}} \left| k_w^{[\alpha]}(z) - k_u^{[\alpha]}(z) e^{-\alpha i \operatorname{Im}(w\overline{u})} \right| |z|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} dA(u), \\ &\leq \int_{\mathbb{C}} C|z|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}} - |w|^{\frac{1}{q}}\right]^2} \left[\left| 1 - \frac{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}} - |u|^{\frac{1}{q}}\right]^2}}{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}} - |w|^{\frac{1}{q}}\right]^2}} \right| + \left| 1 + \frac{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}} - |u|^{\frac{1}{q}}\right]^2}}{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}} - |w|^{\frac{1}{q}}\right]^2}} \right| \right] dA(u). \end{aligned}$$

Now, again from we pick-up from (5.42) as follows:

$$\begin{aligned} \left[\left| 1 - \frac{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}} - |u|^{\frac{1}{q}}\right]^2}}{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}} - |w|^{\frac{1}{q}}\right]^2}} \right| + \left| 1 + \frac{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}} - |u|^{\frac{1}{q}}\right]^2}}{e^{-\frac{\alpha}{2}\left[|z|^{\frac{1}{q}} - |w|^{\frac{1}{q}}\right]^2}} \right| \right] &\leq 2e^{\frac{\alpha}{2}\left[\left[|z|^{\frac{1}{q}} - |w|^{\frac{1}{q}}\right]^2 - \left[|z|^{\frac{1}{q}} - |u|^{\frac{1}{q}}\right]^2\right]}, \\ &\leq 2e^{\frac{\alpha}{2}\left[\left[|z|^{\frac{1}{q}} - |w|^{\frac{1}{q}}\right]^2 - \left[|z|^{\frac{1}{q}} - r^{\frac{1}{q}}\right]^2\right]}, \\ &\leq 2e^{\frac{\alpha}{2}\left[|w|^{\frac{2}{q}} - 2|zw|^{\frac{1}{q}} - r^{\frac{2}{q}} + 2|zr|^{\frac{1}{q}}\right]}, \\ &\leq 2re^{\frac{\alpha}{2}\left[\left(|z|^{\frac{1}{q}} - |w|^{\frac{1}{q}}\right)^2 + |z|^{\frac{2}{q}}\right]}. \end{aligned}$$

With this, it is now clear that we can find a positive constant C depending only on α, such that $H(w,u) \leq Cr$ for all w and u. It follows that

$$\int_{\mathbb{C}} |D_r f(z)| e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} |z|^{\frac{1}{q}-1} dA(z) \leq Cr \int_{\mathbb{C}} |f(u)| e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}} |u|^{\frac{1}{q}-1} dA(u)$$

for all $f \in M_\alpha^\star(1)$. Thus, the norm of D_r satisfies

$$\|D_r\|_{M_\alpha^\star(1)} < Cr, \qquad 0 < r < 1.$$

By the fact that $\|D_r\|_{M_\alpha^\star(\infty)} < Cr$ and $\|D_r\|_{M_\alpha^\star(1)} < Cr$, this imply that for sufficiently small r, $\|D_r\|_{M_\alpha^\star(\infty)} < 1$ and $\|D_r\|_{M_\alpha^\star(1)} < 1$. By the complex interpolation, we have $\|D_r\|_{M_\alpha^\star(p)} < 1$ for all $p \in [1,\infty]$. This shows that T_r is invertible for r being small enough on $M_\alpha^\star(p)$ for all p from $[1,\infty]$. Hence, for $f \in M_\alpha^\star(p)$, we can write $f = T_r g$ with $g = T_r^{-1} f$, and we obtain the atomic decomposition $f(z) = \sum_{w \in r\mathbb{Z}^2} c_w k_w^{[\alpha]}(z)$, with

$$c_w = \frac{\alpha}{q\pi} \int_{S_r+w} g(u) |u|^{\frac{2}{q}-2} e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}} + \alpha i \operatorname{Im}(w\overline{u})} dA(u), \tag{5.43}$$

and with the help of Lemma 5.39 shows that the above sequence $\{c_w\} \in \ell^p$, whenever $g \in M_\alpha^\star(p)$. This completes the proof. $\square$

We will complete the proof of the case $0 < p < 1$ after we have proved the following three lemmas.

Lemma 5.44. *Say $0 < r < 1$ and $0 < p \leq 1$ and $m \in \mathbb{Z}_+$. For any entire functions f we define a sequence,*

$$\left\{ (Sf)_{w,k} : w \in r\mathbb{Z}^2, 0 \leq k \leq m \right\}$$

$$(Sf)_{w,k} = \frac{\alpha}{q\pi} \int_{S_r+w} e^{\alpha i \operatorname{Im} z(\overline{z}-\overline{w}) - \frac{\alpha}{2}|z-w|^2} \frac{(\overline{z}-\overline{w})^k}{\Gamma(qk+1)} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} f(z) |z|^{\frac{2}{q}-2} dA(z).$$

Then S maps $M_\alpha^\star(p)$ boundedly into ℓ^p.

Proof. For any $w \in r\mathbb{Z}^2$, $z \in S_r + w$ and $1 \le k \le m$, we have $|(Sf)_{w,k}|^p$ as follows:

$$
\begin{aligned}
&= \frac{\alpha^p}{(q\pi)^p}\left|\int_{S_r+w} e^{\alpha i\,\mathrm{Im}(\overline{z}-\overline{w})z-\frac{\alpha|z-w|^2}{2}} \frac{(\overline{z}-\overline{w})^k}{\Gamma(qk+1)} f(z)z^{\frac{1}{q}-1}\left[K_q^{[\alpha]}(z-w,w)\right]^{-1} e^{-\frac{\alpha|w|^{\frac{2}{q}}}{2}+\alpha i\,\mathrm{Im}(z-w)\overline{w}} dA(z)\right|^p,\\
&= \frac{\alpha^p}{(q\pi)^p}\left|\int_{S_r+w} e^{-\alpha|z-w|^2-\frac{\alpha|w|^{\frac{2}{q}}}{2}}\left[K_q^{[\alpha]}(z-w,w)\right]^{-1} f(z)z^{\frac{1}{q}-1}\frac{(\overline{z}-\overline{w})^k}{\Gamma(qk+1)}dA(z)\right|^p,\\
&\le \frac{\alpha^p}{(q\pi)^p}\left[e^{-\frac{\alpha}{2}|w|^{\frac{2}{q}}}\int_{S_r+w} e^{-\alpha|z-w|^2}\left|f(z)z^{\frac{1}{q}-1}\left[K_q^{[\alpha]}(z-w,w)\right]^{-1}\frac{(\overline{z}-\overline{w})^k}{\Gamma(qk+1)}\right|dA(z)\right]^p,\\
&\le \frac{\alpha^p}{(q\pi)^p}C\frac{r^{pk}}{(\Gamma(qk+1))^p}\left[e^{-\frac{\alpha}{2}|w|^{\frac{2}{q}}}\int_{S_r+w}\left|f(z)z^{\frac{1}{q}-1}\left[K_q^{[\alpha]}(z-w,w)\right]^{-1}\right|dA(z)\right]^p,\\
&\le C_1 r^{p(2+k)}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}\sup\left\{\left|f(z)z^{\frac{1}{q}-1}\left[K_q^{[\alpha]}(z-w,w)\right]^{-1}\right|^p : z\in S_r+w\right\},\\
&\le C_2 r^{p(2+k)-2}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}\int_{Q_r+w}\left|f(z)z^{\frac{1}{q}-1}\left[K_q^{[\alpha]}(z-w,w)\right]^{-1}\right|^p dA(z),\\
&= C_2 r^{p(2+k)-2}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}\int_{Q_r+w}|f(z)|^p\,|z|^{\frac{p}{q}-p}\left|K_q^{[\alpha]}(z-w,w)\right|^{-p} dA(z),\\
&= C_2 r^{p(2+k)-2}e^{-\frac{p\alpha}{2}|w|^{\frac{2}{q}}}\int_{Q_r+w}|f(z)|^p\,|z|^{\frac{p}{q}-p}\left|E_q(\alpha^q(z\overline{w}-|w|^2)\right|^{-p} dA(z),\\
&\le C_2 r^{p(2+k)-2}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}\int_{Q_r+w}|f(z)|^p\,|z|^{\frac{p}{q}-p}\,e^{-\alpha p\left(|z\overline{w}-|w|^2|^{\frac{1}{q}}\right)}dA(z),\\
&\le C_2 r^{p(2+k)-2}\int_{Q_r+w}|f(z)|^p\,|z|^{\frac{p}{q}-p}\,e^{-\frac{\alpha p}{2}|z|^{\frac{2}{q}}}e^{\frac{\alpha p}{2}|z|^{\frac{2}{q}}}e^{-\frac{\alpha p}{2}|w|^{\frac{2}{q}}}dA(z),\\
&= C_2 r^{p(2+k)-2}\int_{Q_r+w}|f(z)|^p\,|z|^{\frac{p}{q}-p}\,e^{-\frac{\alpha p}{2}|z|^{\frac{2}{q}}}e^{\frac{\alpha p|z|^{\frac{1}{q}}}{2}\left[|z|^{\frac{1}{q}}-|w|^{\frac{1}{q}}\right]}dA(z),\\
&\le C_3 r^{p(2+k)-2}\int_{Q_r+w}|f(z)|^p\,|z|^{\frac{p}{q}-p}\,e^{-\frac{\alpha p}{2}|z|^{\frac{2}{q}}}dA(z).
\end{aligned}
$$

Finally, by letting $C_4 = C_3(m+1)$, then

$$
\begin{aligned}
\sum_{w\in r\mathbb{Z}^2}\sum_{k=0}^{m}\left|(Sf)_{w,k}\right|^p &\le C_4 r^{2(p-1)}\sum_{w\in r\mathbb{Z}^2}\int_{Q_r+w}\left|f(z)z^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}}\right|^p dA(z),\\
&\le C_5 r^{2(p-1)}\int_{\mathbb{C}}\left|f(z)z^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}}\right|^p dA(z),
\end{aligned}
$$

which proves the desired result. $\square$

Lemma 5.45. *Suppose $r \in (0,1)$ and $p \in (0,1]$ and m is a non-negative integer. For every sequence*

$$c = \{c_{w,k} : w \in r\mathbb{Z}^2, 0 \le k \le m\},$$

define a function Tc by

$$Tc(z) = \sum_{w \in r\mathbb{Z}^2} \sum_{k=0}^{m} c_{w,k} \left[\alpha^q (z-w)\right]^k k_w^{[\alpha]}(z),$$

then T is a bounded linear operator from ℓ^p into $M_\alpha^\star(p)$.

Proof. We have following chain of inequalities:

$$\begin{aligned}
|Tc(z)|^p &\le \sum_{w \in r\mathbb{Z}^2} \sum_{k=0}^{m} |c_{w,k}|^p |[\alpha^q (z-w)]|^{pk} \left|k_w^{[\alpha]}(z)\right|^p, \\
\|Tc\|^p_{M_\alpha^\star(p)} &\le \sum_{w \in r\mathbb{Z}^2} \sum_{k=0}^{m} |c_{w,k}|^p \alpha^{qpk} \int_{\mathbb{C}} [|z|+|w|]^{pk} \left|k_w^{[\alpha]}(z)\right|^p |z|^{\frac{p}{q}-p} e^{-\frac{\alpha p}{2}|z|^{\frac{2}{q}}} dA(z), \\
&\le C \sum_{w \in r\mathbb{Z}^2} \sum_{k=0}^{m} |c_{w,k}|^p \alpha^{qpk} |2w|^{pk}.
\end{aligned}$$

This proves the desired result. □

Lemma 5.46. *Let r_0 be the number from* THEOREM *5.43 in the case $p = \infty$. Suppose $0 < r < r_0$ and $p \in (0,1]$. Then every monomial z^k can be represented as*

$$z^k = \sum_{w \in r\mathbb{Z}^2} c_w k_w^{[\alpha]}(z),$$

where $\{c_w\} \in \ell^p$.

Proof. Fix $\rho \in (r, r_0)$. By the already-proved case $p = \infty$ of THEOREM 5.43, every monomial z^k can be represented as

$$z^k = \sum_{w \in r\mathbb{Z}^2} c_w k_w^{[\alpha]}(z),$$

where $\{c_w\} \in \ell^\infty$. For $w \in \rho\mathbb{Z}^2$, we can write $w = \rho(m+in)$ for some integers m and n. Since,

$$k_w^{[\alpha]}(z) = \frac{K_q^{[\alpha]}(z,w)}{\sqrt{K_q^{[\alpha]}(w,w)}},$$

we have for $w' = r(m+in)$ that

$$k_w^{[\alpha]}\left(\left(\frac{r}{\rho}\right)z\right) = \frac{K_q^{[\alpha]}(z,w')}{\sqrt{K_q^{[\alpha]}(w',w')}}\frac{\sqrt{K_q^{[\alpha]}(w',w')}}{\sqrt{K_q^{[\alpha]}(w,w)}},$$

it follows that,

$$\left(\frac{r}{\rho}z\right)^k = \sum_{w' \in r\mathbb{Z}^2} c'_{w'} k_{w'}^{[\alpha]}(z),$$

where

$$c'_{w'} = \frac{K_q^{[\alpha]}(w',w')}{K_q^{[\alpha]}(w,w)} = c_w \frac{E_q(\alpha^q |w'|^2)}{E_q(\alpha^q |w|^2)} = c_w \frac{E_q\left(\alpha^q r^2(m^2+n^2)\right)}{E_q\left(\alpha^q \rho^2(m^2+n^2)\right)}.$$

Recall that $r < \rho$, therefore, this sequence is in ℓ^p and this shows the atomic decomposition of monomials. □

Case 2: $p = \infty$ We will now finish the proof for the atomic decomposition for $M_\alpha^\star(p)$ for $0 < p < 1$. Fix a sufficiently small $r \in (0,1)$ and let m be the integer part of $2(1-p)/p$ and let S and T be the operators defined in previous two lemmas. We have then,

$$(I - TS)\, f(z) = \frac{\alpha}{q\pi} \sum_{w \in r\mathbb{Z}^2} \int_{S_r + w} G(z,w,u) f(u) e^{-\frac{\alpha p}{2}|u|^{\frac{2}{q}}} |u|^{\frac{2}{q}-2} dA(u)$$

where

$$G(z,w,u) = k_w^{[\alpha]}(z)e^{\alpha i \operatorname{Im} u(\overline{u}-\overline{w})-\frac{\alpha}{2}|u-w|^2} K_q^{[\alpha]}(z-w, u-w)$$
$$- k_w^{[\alpha]}(z)e^{\alpha i \operatorname{Im} u(\overline{u}-\overline{w})-\frac{\alpha}{2}|u-w|^2}\left[\sum_{k=0}^{m}\frac{\left[\alpha^q(z-w)(\overline{u-w})\right]^k}{\Gamma(qk+1)}\right],$$

which is same to say that

$$G(z,w,u) = k_w^{[\alpha]}(z)e^{\alpha i \operatorname{Im} u(\overline{u}-\overline{w})-\frac{\alpha}{2}|u-w|^2}\left[\sum_{k=m+1}^{\infty}\frac{\left[\alpha^q(z-w)(\overline{u-w})\right]^k}{\Gamma(qk+1)}\right],$$

and therefore, for $u \in S_r + w$, we have $|u-w| < r$, therefore,

$$|G| \le \left|k_w^{[\alpha]}(z)\right| \sum_{k=m+1}^{\infty}\frac{\left[\alpha^q\,|z-w|\,r\right]^k}{\Gamma(qk+1)},$$

and so by Holder's inequality $|(I-TS)f(z)|^p$ is less than or equal to

$$\frac{\alpha^p}{(q\pi)^p}\sum_{w\in r\mathbb{Z}^2}\left|k_w^{[\alpha]}(z)\right|^p\left[\sum_{k=m+1}^{\infty}\frac{\left[\alpha^q|z-w|r\right]^k}{\Gamma(qk+1)}\right]^p\left[\int_{S_r+w}\left|f(u)u^{\frac{1}{q}-1}e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}}\right|dA(u)\right]^p.$$

It follows from this and Fubini's theorem that

$$\int_{\mathbb{C}}\left|(I-TS)f(z)e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}}z^{\frac{1}{q}-1}\right|^p dA(z)$$

less than or equal to less than or equal to

$$\frac{\alpha^p}{(q\pi)^p}\sum_{w\in r\mathbb{Z}^2}C(w)\left[\int_{S_r+w}|f(u)||u|^{\frac{1}{q}}e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}}dA(u)\right]^p,$$

where

$$C(w) = \int_{\mathbb{C}}\left|k_w^{[\alpha]}(z)\right|^p\left[\sum_{k=m+1}^{\infty}\frac{\left[\alpha^q|z-w|r\right]^k}{\Gamma(qk+1)}\right]^p dA(z),$$

$$\leq \frac{r^{p(m+1)}}{|E_q(\alpha^q|w|^2)|^{\frac{p}{2}}} \int_{\mathbb{C}} \left[\sum_{k=m+1}^{\infty} \frac{[\alpha^q|z-w|]^k}{\Gamma(qk+1)} \right]^p dA(z),$$

$$\leq \frac{r^{p(m+1)}}{|E_q(\alpha^q|w|^2)|^{\frac{p}{2}}} \int_{\mathbb{C}} |K_q^{[\alpha](z,w)}|^p \left[\sum_{k=m+1}^{\infty} \frac{\alpha^q\left[|z|+|w|\right]^k}{\Gamma(qk+1)} \right]^p dA(z),$$

$$= r^{p(m+1)} |w|^{p-\frac{p}{q}} e^{\frac{\alpha}{2}|w|^{\frac{2}{q}}} \left[\sum_{k=m+1}^{\infty} \frac{\alpha^q\left[2|w|\right]^k}{\Gamma(qk+1)} \right]^p.$$

So, there is a constant $C > 0$ such that

$$\int_{\mathbb{C}} \left| (I - TS) f(z) z^{\frac{1}{q}-1} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} \right|^p dA(z)$$

is less than or equal to

$$C r^{pm+p} \sum_{w \in r\mathbb{Z}^2} \left[\int_{S_r+w} |f(u)| e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}} |u|^{\frac{1}{q}-1} dA(u) \right]^p.$$

On the other hand,

$$\left[\int_{S_r+w} |f(u)| e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}} |u|^{\frac{1}{q}-1} dA(u) \right]^p \leq r^{2p} \sup_{u \in S_r+w} \left| f(u) |u|^{\frac{1}{q}-1} e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}} \right|^p,$$

and an application of LEMMA 5.39 produces another constant $C > 0$ which is independent of $r \in (0,1)$ such that

$$\left[\int_{S_r+w} |f(u)| e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}} |u|^{\frac{1}{q}-1} dA(u) \right]^p \leq C r^{2p-2} \int_{Q_r+w} \left[|f(u)| |u|^{\frac{1}{q}-1} e^{-\frac{\alpha}{2}|u|^{\frac{2}{q}}} \right]^p dA(u).$$

Thus,

$$\int_{\mathbb{C}} \left| (I-TS)\, f(z) |z|^{\frac{1}{q}-1} e^{-\frac{\alpha}{2}|z|^{\frac{2}{q}}} \right|^p dA(z) \leq C r^{pm+3p-2} \sum_{w \in r\mathbb{Z}^2} \int_{\mathbb{C}} |f(u)|^p |u|^{\frac{p}{q}-p} e^{-\frac{\alpha p}{2}|u|^{\frac{2}{q}}} dA(u).$$

So, we can find another constant $C > 0$ independent of r such that

$$\|I - TS\|_{M_\alpha^\star(p)} \leq C r^{m+3-\frac{2}{p}},$$

where $0 < r < 1$ and obviously $m + 3 - \frac{2}{p} > 0$. We see that there exists some $r_0 \in (0, 1)$ such that $\|I - TS\|_{M_\alpha^\star(p)} < 1$ whenever $r \in (0, r_0)$. Thus, TS is an invertible operator on $M_\alpha^\star(p)$ where $r \in (0, r_0)$. Since, for all $r \in (0, r_0)$, the operator T is onto and thus every function $f \in M_\alpha^\star(p)$ can be written as

$$f(z) = \sum_{w \in r\mathbb{Z}^2} \sum_{k=0}^{m} c_{w,k} (z - w)^k k_w^{[\alpha]}(z). \tag{5.44}$$

Now, the coefficients $c_{w,k}$ in (5.44) above all depends on f linearly and we have

$$\sum_{w \in r\mathbb{Z}^2} \sum_{k=0}^{m} |c_{w,k}|^p \leq C \|f\|^p_{M_\alpha^\star(p)},$$

where C is a positive constant independent of f. Given any $\delta > 0$ and $\forall\, r \in (0, r_0)$, it follows from LEMMA 5.46 that there exist coefficient $c_{w,k}$, $0 \leq k \leq m$, $w \in r\mathbb{Z}^2$ and $|w| \leq N$ such that

$$\left\| z^k - \sum_{u \in r\mathbb{Z}^2, |u| \leq N} c'_{u,k} k_w^{[\alpha]}(z) \right\|_{M_\alpha^\star(p)} < \delta,$$

for all $0 \leq k \leq m$. By a change of variables,

$$\left\| (z - w)^k k_w^{[\alpha]}(z) - \sum_{u \in r\mathbb{Z}^2} c'_{u,k} k_u^{[\alpha]}(z - w) k_w^{[\alpha]}(z) \right\|_{M_\alpha^\star(p)} < \delta,$$

for all $0 \leq k \leq m$ and $w \in r\mathbb{Z}^2$. The above inequality can be modified as follows:

$$\left\| (z - w)^k k_w^{[\alpha]}(z) - \sum_{u \in r\mathbb{Z}^2} c'_{u,k} e^{\alpha i \operatorname{Im} w\bar{u}} k_{u+w}^{[\alpha]}(z) \right\|_{M_\alpha^\star(p)} < \delta.$$

Now, define an operator A_r on $M^{\star}_{\alpha}(p)$ as follows:

$$A_r f(z) = \sum_{w \in r\mathbb{Z}^2} c_{w,k} \sum_{u \in r\mathbb{Z}^2, |u| \leq N} c'_{u,k} e^{\alpha i \operatorname{Im}(\overline{u}w)} k^{[\boldsymbol{\alpha}]}_{u+w}(z)$$

It then follows from Hölder's inequality that,

$$\|f - A_r f\|^p_{M^{\star}_{\alpha}(p)} \leq \sum_{w \in r\mathbb{Z}^2, 0 \leq k \leq m} |c_{w,k}|^p \delta^p \leq C\delta^p \|f\|^p_{ML^p}$$

for all $f \in M^{\star}_{\alpha}(p)$. Now, choosing δ in such a manner that $C\delta^p < 1$, then $\|I - A_r\|_{M^{\star}_{\alpha}(p)} < 1$ and so the operator A_r is surjective on $M^{\star}_{\alpha}(p)$. Since $w + u \in r\mathbb{Z}^2$ whenever $w \in r\mathbb{Z}^2$ and $u \in r\mathbb{Z}^2$, the proof of *atomic decomposition* of $M^{\star}_{\alpha}(p)$ is now complete.

5.6 Maximum Principle

We provide the maximum principle for the $M^{\star}_{\alpha}(p)$ in this section.

Theorem 5.47. *Let $p \geq 1$ and $\alpha > 0$ and suppose $|f(z)| \leq |g(z)|$ over the region $|z| \geq R$, then following is true:*

$$\int_{\mathbb{C}} |f(z)|^p d\mathbf{J}_{\alpha}(z)^{[\mathbf{p}]} \leq \int_{\mathbb{C}} |g(z)|^p d\mathbf{J}_{\alpha}(z)^{[\mathbf{p}]}.$$

Proof. Without loss of generality, we will assume that $g \in M^{\star}_{\alpha}(p)$. Otherwise, the desired result is obvious. Under this assumption, we also have

$$\int_{|z| \geq R} |f|^p d\mathbf{J}_{\alpha}(z)^{[\mathbf{2}]} \leq \int_{|z| \geq R} |g|^p d\mathbf{J}_{\alpha}(z)^{[\mathbf{2}]} < \infty,$$

and therefore, $f \in M^{\star}_{\alpha}(p)$ also. We consider following for the positive r and any complex function F,

$$I(r, F) = \int_0^{2\pi} F\left(re^{i\theta}\right) d\theta.$$

Let some positive R is fixed and we proceed by assuming $|f(z)|\leq |g(z)|$ over the region $R < |z|< \infty$. For $r \in (0, R)$ and $\rho \in (R, \infty)$, we will compare $I(r,_f \Delta_g)$ with $I(\rho, -_f\Delta_g)$, where $_f\Delta_g = |f|^p - |g|^p$. Note that, considering $w(z) = \frac{f(z)}{g(z)}$ over the region $|z|\in (R, \infty)$, $\omega(z)$ is analytic with its modulus strictly less than 1. Even if, the modulus of $\omega(z)$ is 1 for some z_0 where $|z_0|\in (R, \infty)$, then by the classical maximum modulus principle, the analytic function $\omega(z)$ is constant. In this case, f and g differs by a constant multiple in the whole complex plane, from which the desired result clearly follows.

Pick a point $\zeta(\rho) \in \mathbb{C}$ such that its modulus is ρ where $\rho \in (R, \infty)$ and along with this following is also satisfied:

$$|w(\zeta(\rho))|= \max \{|\omega(z)|: |z|= \rho\}.$$

Thus, $0 < |w(\zeta(\rho))|< 1$ for all $\rho \in (R, \infty)$. Here, we are not assuming the $f = 0$, otherwise the result follows trivially. For the sake of simplified notation, say $\omega_\rho = \omega(\zeta(\rho))$. Now consider following two inequalities for $p \geq 1$ and both x and y at-least greater than 0:

$$x^p - y^p \leq px^{p-1}(x - y) \tag{5.45}$$

and

$$py^{p-1}(x - y) \leq x^p - y^p. \tag{5.46}$$

We will use both of the above inequalities to discuss the comparison between $I(r, \Delta)$ with $I(\rho, -\Delta)$ as follows. We use (5.45) to have following:

$$\begin{aligned} I(r,_f \Delta_g) &= \int_0^{2\pi} |f(re^{i\theta})|^p - |g(re^{i\theta})|^p d\theta, \\ &\leq \int_0^{2\pi} p|f(re^{i\theta})|^{p-1}\left[|f(re^{i\theta})| - |\omega_\rho g(re^{i\theta})|\right] d\theta, \\ &\leq \int_0^{2\pi} p|f(re^{i\theta})|^{p-1}\left[|f(re^{i\theta}) - \omega_\rho g(re^{i\theta})|\right] d\theta, \end{aligned} \tag{5.47}$$

$$
\begin{aligned}
(5.48) \qquad &\leq \int_0^{2\pi} p|f(\rho e^{i\theta})|^{p-1}\left[|f(\rho e^{i\theta}) - \omega_\rho g(\rho e^{i\theta})|\right] d\theta \\
&= \int_0^{2\pi} p\frac{|\omega|^{p-1}}{1-|\omega|^p}\left[|g(\rho e^{i\theta})|^p - |f(\rho e^{i\theta})|^p\right]|\omega - \omega_\rho| d\theta,
\end{aligned}
$$

We used the property of subharmonicity & increasing in r nature of the argument in the integral in (5.47) to (5.48). We use (5.46), to realize that following is true:

$$
p\frac{|\omega|^{p-1}}{1-|\omega|^p} \leq \frac{1}{1-|\omega|} = \frac{1+\omega}{1-|\omega|^2} < \frac{2}{1-|\omega|^2}.
$$

Therefore, in conclusion we have following:

$$
I(r,{}_f\Delta_g) \leq \int_0^{2\pi} \frac{2}{1-|\omega|^2}|\omega - \omega_\rho|\left[|g(\rho e^{i\theta})|^p - |f(\rho e^{i\theta})|^p\right] d\theta.
$$

Consider $\gamma(\rho)$ as a following quantity:

$$
\gamma(\rho) = \max\left\{\frac{|\omega(z) - \omega(\zeta(\rho))|}{1-|\omega(z)|^2} : |z| = \rho, \rho \in (R,\infty)\right\}.
$$

Therefore following is achieved $\forall\ 0 \leq r \leq R < \rho < \infty$,

$$
\begin{aligned}
&I(r,{}_f\Delta_g) \leq 2\gamma(\rho) I(\rho, -{}_f\Delta_g), \\
(5.49) \qquad &\int_0^{2\pi}\left[|f(re^{i\theta})|^p - |g(re^{i\theta})|^p\right] d\theta \leq \int_0^{2\pi}\left[|g(\rho e^{i\theta})|^p - |f(\rho e^{i\theta})|^p\right] d\theta.
\end{aligned}
$$

We will integrate both sides of (5.49) against the measure $\frac{\alpha r^{\frac{2}{q}-2}}{\pi q} e^{-\alpha r^{\frac{2}{q}}} r dr$ over the limits of $[0, R]$ as follows:

$$
\begin{aligned}
\int_{|z|\leq R} {}_f\Delta_g d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} &\leq \left(\int_0^R \frac{\alpha r^{\frac{2}{q}-2}}{\pi q} e^{-\alpha r^{\frac{2}{q}}} r dr\right)\gamma(\rho)\int_{|z|\leq R} -{}_f\Delta_g d\mathbf{J}_\alpha(z)^{[\mathbf{2}]}, \\
&= \frac{q\left(1 - e^{-\alpha R^{\frac{2}{q}}}\right)}{2\alpha}\gamma(\rho)\int_{|z|\leq R} -{}_f\Delta_g d\mathbf{J}_\alpha(z)^{[\mathbf{2}]}.
\end{aligned}
$$

This implies that

$$\frac{1}{\gamma(\rho)}\left(\int_{|z|\le R} {}_f\Delta_g d\mathbf{J}_\alpha(z)^{[\mathbf{2}]}\right) \le \frac{q\left(1-e^{-\alpha R^{\frac{2}{q}}}\right)}{2\alpha}\int_{|z|\le R} -{}_f\Delta_g d\mathbf{J}_\alpha(z)^{[\mathbf{2}]}. \tag{5.50}$$

We will integrate (5.50) against the measure $\dfrac{\alpha\rho^{\frac{2}{q}-2}}{\pi q}e^{-\alpha\rho^{\frac{2}{q}}}\rho d\rho$ to have following:

$$\int_R^\infty \frac{\rho^{\frac{2}{q}-2}}{\gamma(\rho)}e^{-\alpha\rho^{\frac{2}{q}}}\rho d\rho \int_{|z|\le R} {}_f\Delta_g d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} \le \frac{\left(1-e^{-\alpha R^{\frac{2}{q}}}\right)}{2\pi}\int_{|z|\ge R} -{}_f\Delta_g d\mathbf{J}_\alpha(z)^{[\mathbf{2}]}.$$

In conclusion, we get following:

$$\int_{|z|\le R} {}_f\Delta_g d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} \le C_R \int_{|z|\ge R} -{}_f\Delta_g d\mathbf{J}_\alpha(z)^{[\mathbf{2}]},$$

where

$$C_R = \frac{q\left(1-e^{-\alpha R^{\frac{2}{q}}}\right)}{2\alpha \int_R^\infty \dfrac{\rho^{\frac{2}{q}-2}e^{-\alpha\rho^{\frac{2}{q}}}}{\gamma(\rho)} d\rho}. \tag{5.51}$$

We will show $C_R < 1$ (in LEMMA 5.48) and as of now we are assuming and proceeding further as follows:

$$\begin{aligned}
\int_{\mathbb{C}} {}_f\Delta_g d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} &= \int_{\mathbb{C}} |f|^p - |g|^p d\mathbf{J}_\alpha(z)^{[\mathbf{2}]},\\
&= \int_{|z|\le R} \left[|f|^p - |g|^p\right] d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} + \int_{|z|\ge R} \left[|f|^p - |g|^p\right] d\mathbf{J}_\alpha(z)^{[\mathbf{2}]},\\
&\le C_R \int_{|z|\ge R} \left[|g|^p - |f|^p\right] d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} + \int_{|z|\ge R} \left[|f|^p - |g|^p\right] d\mathbf{J}_\alpha(z)^{[\mathbf{2}]},\\
&\le \int_{|z|\ge R} \left[|g|^p - |f|^p\right] d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} + \int_{|z|\ge R} \left[|f|^p - |g|^p\right] d\mathbf{J}_\alpha(z)^{[\mathbf{2}]},\\
&= 0,\\
\implies \int_{\mathbb{C}} |f|^p d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} &\le \int_{\mathbb{C}} |g|^p d\mathbf{J}_\alpha(z)^{[\mathbf{2}]}.
\end{aligned}$$

$\square$

Lemma 5.48. *The value of* C_R *in* (5.51) *is less than* 1.

Proof. Recall the Mobius pseudo-hyperbolic metric in the unit disc $\mathbb{D}$ as follows:

$$\mathfrak{m}(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|. \tag{5.52}$$

After recalling $\mathfrak{m}(z,w)$ in (5.52), we can have following setting:

$$\frac{|u-v|}{1-|u|^2} = \frac{\mathfrak{m}(u,v)}{\sqrt{1-\mathfrak{m}(u,v)^2}}\frac{\sqrt{1-|v|^2}}{\sqrt{1-|u|^2}}. \tag{5.53}$$

Replacing u by $\omega(z)$ and v by $\omega(\zeta(\rho))$ in (5.53) to have following consequences:

$$\begin{aligned}
\frac{|w(z)-\omega(\zeta(\rho))|}{1-|w(z)|^2} &= \frac{\mathfrak{m}(w(z),\omega(\zeta(\rho)))}{\sqrt{1-\mathfrak{m}(w(z),\omega(\zeta(\rho)))^2}}\frac{\sqrt{1-|\omega(\zeta(\rho))|^2}}{\sqrt{1-|w(z)|^2}}, \\
&\leq \frac{\mathfrak{m}(w(z),\omega(\zeta(\rho)))}{\sqrt{1-\mathfrak{m}(w(z),\omega(\zeta(\rho)))^2}}.
\end{aligned}$$

It follows that for $\rho \in (R,\infty)$

$$\gamma(\rho) \leq \sup_{|z|=\rho} \frac{\mathfrak{m}(w(z),\omega(\zeta(\rho)))}{\sqrt{1-\mathfrak{m}(w(z),\omega(\zeta(\rho)))^2}}.$$

If we take $H(z) = \omega(R/z)$, then we realize that it has removable singularity at 0 and is bounded in the neighbourhood of 0. Also, it maps $0 < |z| < 1$ analytically to the unit disc $\mathbb{D}$. Thus, we can picture this map as a self-analytic map over $\mathbb{D}$. Hence, with the application of Schwarz lemma, we have following deduction:

$$\mathfrak{m}(H(z),H(w)) \leq \mathfrak{m}(z,w),$$

which is why,

$$\mathfrak{m}((w(z),\omega(\zeta(\rho)))) \leq \mathfrak{m}(H(R/z), H(R/\omega(\zeta(\rho)))) \leq \mathfrak{m}(R/z, R/\omega(\zeta(\rho))).$$

for all $|z|=\rho$. Therefore,

$$\gamma(\rho) \leq \sup_{|z|=\rho} \frac{\mathfrak{m}(R/z, R/\omega(\zeta(\rho)))}{\sqrt{1-\mathfrak{m}(R/z, R/\omega(\zeta(\rho)))^2}}.$$

When we take the symmetry of $\mathbb{D}$ into account,

$$\sup_{|z|=\rho} \mathfrak{m}(R/z, R/\omega(\zeta(\rho))) = \mathfrak{m}(-R/\omega(\zeta(\rho)), R/\omega(\zeta(\rho))) = \frac{2R\rho}{\rho^2+R^2},$$

therefore

$$\begin{aligned}
\gamma(\rho) &\leq \frac{2R\rho}{\rho^2-R^2} & \textit{for } R<\rho<\infty, \\
\Longrightarrow \frac{1}{\gamma(\rho)} &\geq \frac{\rho^2-R^2}{2R\rho}, \\
\frac{\rho^{\frac{2}{q}-2}e^{-\alpha\rho^{\frac{2}{q}}}\rho d\rho}{\gamma(\rho)} &\geq \frac{(\rho^2-R^2)\rho^{\frac{2}{q}-2}e^{-\alpha\rho^{\frac{2}{q}}}\rho d\rho}{2R\rho}, \\
\int_R^\infty \frac{\rho^{\frac{2}{q}-2}e^{-\alpha\rho^{\frac{2}{q}}}\rho d\rho}{\gamma(\rho)} &\geq \int_R^\infty \frac{(\rho^2-R^2)\rho^{\frac{2}{q}-2}e^{-\alpha\rho^{\frac{2}{q}}}\rho d\rho}{2R\rho}, \\
\Longrightarrow \frac{1}{\int_R^\infty \frac{\rho^{\frac{2}{q}-2}e^{-\alpha\rho^{\frac{2}{q}}}\rho d\rho}{\gamma(\rho)}} &\leq \frac{2R}{\int_R^\infty (\rho^2-R^2)\rho^{\frac{2}{q}-2}e^{-\alpha\rho^{\frac{2}{q}}}d\rho}, \\
\Longrightarrow C_R &\leq \frac{Rq}{\alpha}\frac{(1-e^{\alpha R^{\frac{2}{q}}})}{\int_R^\infty (\rho^2-R^2)\rho^{\frac{2}{q}-2}e^{-\alpha\rho^{\frac{2}{q}}}d\rho}
\end{aligned}$$

as $R \longrightarrow 0^+$. Therefore, we have $C_R < 1$. □

5.7 Translation

Suppose $a \in \mathbb{C}$, then we define following three analytic self-maps of $\mathbb{C}$:

$$t_a(z) := z + a,$$

$$\tau_a(z) := z - a,$$

$$\phi_a(z) := a - z.$$

We have following attributes for t_a, τ_a and ϕ_a. The map t_a is called the translation by a, and it is clear that $\tau_a = t_{-a} = t_a^{-1}$. The map ϕ_a is the composition of the translation t_a with the reflection $z \to -z$. Note that ϕ_a is its own inverse. From (5.7) and (5.8), following easily follows:

$$\int_{\mathbb{C}} f \circ \tau_w(z) d\mathbf{J}_\alpha(w)^{[\mathbf{2}]} = \int_{\mathbb{C}} f(w) \frac{1}{E_q(\alpha^q |z-w|^2)} d\mathbf{J}_\alpha(w)^{[\mathbf{2}]},$$
$$\int_{\mathbb{C}} f \circ t_w(z) d\mathbf{J}_\alpha(w)^{[\mathbf{2}]} = \int_{\mathbb{C}} f(w) \frac{K_q^{[\boldsymbol{\alpha}]}(2z, w)}{E_q(\alpha^q |w-z|^2)} d\mathbf{J}_\alpha(w)^{[\mathbf{2}]}.$$

Corollary 5.49. *We have following calculations:*

$$\int_{\mathbb{C}} f \circ \tau_a(z) \left| k_a^{[\boldsymbol{\alpha}]}(z) \right|^2 d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} = \int_{\mathbb{C}} f(z) d\mathbf{J}_\alpha(z), \tag{5.54}$$

$$\int_{\mathbb{C}} f \circ t_a(z) \left| k_a^{[\boldsymbol{\alpha}]}(z) \right|^2 d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} = \int_{\mathbb{C}} f(z+2a) d\mathbf{J}_\alpha(z). \tag{5.55}$$

Proof. We give the proof of (5.54) as follows:

$$\begin{aligned}
\int_{\mathbb{C}} f \circ \tau_a(z) \left| k_a^{[\boldsymbol{\alpha}]}(z) \right|^2 d\mathbf{J}_\alpha(z)^{[\mathbf{2}]} &= \int_{\mathbb{C}} f(z-a) \frac{\left| K_q^{[\boldsymbol{\alpha}]}(z,a) \right|^2}{K_q^{[\boldsymbol{\alpha}]}(a,a)} d\mathbf{J}_\alpha(z)^{[\mathbf{2}]}, \\
&= \frac{1}{K_q^{[\boldsymbol{\alpha}]}(a,a)} \int_{\mathbb{C}} f(z-a) K_q^{[\boldsymbol{\alpha}]}(z,a) K_q^{[\boldsymbol{\alpha}]}(a,z) d\mathbf{J}_\alpha(z), \\
&= f(0),
\end{aligned}$$

$$= \int_{\mathbb{C}} f(z) d\mathbf{J}_\alpha(z)^{[\mathbf{2}]}.$$

In similar ways (5.55) can be proved. $\square$

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